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QUASI-PERIODIC SOLUTIONS TO THE RICCATI DIFFERENTIAL EQUATION

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A b s t r a c t: In this paper we give some conditions for existence of quasi-periodic solutions to the Riccati differential equation and find this solution in a case of a constant quasi-period.

Key words with abbreviation: differential equation (DE); quasi-period (QP); quasi-periodic solution (QPS); quasi-periodic coefficient (QPC); equation (eq.)

1. PRELIMINARY

Let the Riccati differential equation

$$y'(x)+f(x)y(x)+g(x)y^2(x)+h(x)=0 \quad (g(x)\neq 0) \tag{1}$$

be given. We want to find QPS $y=y(x)$ for (1), i.e. to find the solution that satisfies the relation

$$y(x+\omega)=\lambda(x,\omega(x))y(x)=\lambda(x)y(x)\,,\ x,x+\omega\in D_y \tag{2}$$

where $\omega=\omega(x)$ is QP and $\lambda=\lambda(x)$ is QPC for the function $y=y(x)$.

The following theorem holds.

Theorem 1.1. If DE (1) has QPS $y=y(x)$ with QP $\omega=\omega(x)$ and QPC $\lambda(x)$, then it is reduced to the algebraic equation with respect to the QPS y

$$\left(-\frac{1}{t'}\lambda(x)g(x)+\lambda^2(x)g(t)\right)y^2(x)+\left(\frac{\lambda'(x)}{t'}-\frac{\lambda(x)}{t'}f(x)+\lambda(x)f(t)\right)y(x)+$$

$$+\left(-\frac{\lambda(x)}{t'}h(x)+h(t)\right)=0, t=x+\omega(x). \tag{3}$$

Proof. Under the conditions of the theorem we have the system

$$\left.\begin{aligned} &y'(x)+f(x)y(x)+g(x)y^2(x)+h(x)=0\\ &y'(t)+f(t)y(t)+g(t)y^2(t)+h(t)_{/t=x+\omega}=0\\ &y(t)=\lambda(x)y(x)\\ &\frac{d}{dx}y(t)=\lambda'(x)y(x)+\lambda(x)y'(x) \end{aligned}\right\} \tag{4}$$

from where

$$y'(x)=-f(x)y(x)-g(x)y^2(x)-h(x) \tag{5}$$

and

$$y'(t)=\frac{1}{t'}\left(\lambda'(x)y(x)+\lambda(x)y'(x)\right). \tag{6}$$

Subtituting (5) and (6) in the second equation of the system (4), after short transfomations, we obtain (3).

Remark 1.1. In general, solving equation (3) is not a simple problem and we can solve it only in some special cases. So, here we consider QPS for (1) with a constant QP and a constant QPC.

2. QUASIPERIODIC SOLUTIONS WITH CONSTANT QUASI-PEROD AND CONSTANT QUASIPERIODIC COEFFICIENT

Theorem 2.1. Let DE (1) have QPS $y=y(x)$ with a constant QP $\omega=\boldsymbol{c}$ and a constant QPC $\lambda>\mathbf{0}$. Then it is reduced, with respect to QPS *y*, to the equation:

$$\lambda\big(-g(x)+\lambda g(t)\big)y^2(x)+\lambda\big(-f(x)+f(t)\big)y(x)+\big(-\lambda h(x)+h(t)\big)=0\,/_{t=x+c}$$

(7)

Proof. Substituting in eq. (3) $\omega = c,\ \ \omega' - 0,\ \ t = x + c,\ \ t' = 1$ we obtain eq. (7).

Theorem 2.2. Let

1° DF (1) have QPS $y_1(x)$ with a constant QP $\omega = \boldsymbol{c}$ and a constant QPC $\lambda > 0$;

2° the coefficients $f(x), g(x), h(x)$ are QPF with the same QP $\omega = \boldsymbol{c}$ and QPC $\lambda_1 = 1, \lambda_2 = \frac{1}{\lambda}, \lambda_3 = \lambda$ respectively;

3° the general solution for the equation

$$z' - (f(x) + 2g(x)y_1(x))z - g(x) = 0 \tag{8}$$

is QPF with QP $\omega = c$ and QPC $\frac{1}{\lambda}$.

Then every solution for DE (1) is QPF with QP $\omega = c$ and QPC $\lambda > \mathbf{0}$.

Proof. Under the given conitions follows that the general solutions for DE (1) is $y(x) = y_1(x) + \frac{1}{z(x)}$, where $z = z(x)$ [1] is QPS for the eq. (8). Since the coefficients $f(x), g(x), h(x)$ satisfy the relations $y(t) = f(x),\ \ g(t) = \frac{1}{\lambda} g(x),$ $h(i) = \lambda h(x) /_{t=x+c}$, we get

$$y(t) = y_1(t) + \frac{1}{z(t)} = \lambda\left(y_1(x) + \frac{1}{z(x)} \right) = \lambda y(x)\,,$$

and

$$y'(t) + f(t)y(t) + g(t)y^2(t) + h(t)_{/t=x+c} =$$

$$= \lambda y'(x) + f(x)\cdot\lambda y(x) + \frac{1}{\lambda} g(x)\cdot\lambda^2 y^2(x) + \lambda h(x) =$$

$$= \lambda\left(y'(x) + f(x)y(x) + g(x)y^2(x) + h(x)\right) = \lambda\cdot 0 = 0\,. \qquad \square$$

Remark 2.1 If DE (1) has QPS $y_1(x)$ then its general solution is

$$y(x) = y_1(x) + \frac{1}{z(x)} = y_1(x) + \frac{1}{Ca(x) + b(x)}\,,$$

where $a(x) = e^{\int (f(x) + 2g(x)y_1(x))dx}$, $b(x) = a(x)\int \frac{g(x)}{a(x)}dx$.

Example 2.1. Let

$$y' + (3 + ctgx)y - \frac{e^x}{\sin x} y^2 - e^{-x}(\sin x + 2\cos x) = 0\,.$$

This equation has a particular solution $y_1 = e^{-x}\sin x$ that is QPF with a QP $\omega = 2\pi$ and QPC $\lambda = e^{-2\pi}$. Since the coefficients $f(x)$, $g(x)$, $h(x)$ are a QPF with the same QP $\omega = 2\pi$ and QPC 1, $e^{2\pi} = \frac{1}{\lambda}$, $e^{-2\pi}$ respectively, according to the Theorem 2.2. every solution for the given equation is QPF. Indeed, its general solution is $y = e^{-x}\left(\sin x + \frac{1}{C\sin x + \cos x}\right)$, that is QPF with QP $\omega = 2\pi$ and QPC $\lambda = e^{-2\pi}$.

Theorem 2.3. If

1° DE (1) has two QPS $y_1(x)$ and $y_2(x)$, with QP $\omega = c$ and QPC $\lambda > 0$,

2° the coefficients $f(x), g(x), h(x)$ in DE (1) are QPF with QP $\omega = c$ and QPC $\lambda_1 = 1, \lambda_2 = \frac{1}{\lambda}, \lambda_3 = \lambda$ respectively, 3° for $\varphi(x) = e^{-\int g(x)(y_1(x)-y_2(x))dx}$, $z(x) = \dfrac{C\varphi(x)-1}{C\varphi(x)} \dfrac{1}{y_2(x)-y_1(x)}$ is

a general QPS for (8) with QP $\omega = c$ and QPC $\dfrac{1}{\lambda}$,

then every solution for the DE (1) is QPF with QP $\omega = c$ and QPC λ.

Proof. It can be proved in a similar manner as the previous theorem. □

Remark 2.2. If the DE (1) has two QPS $y_1 = y_1(x)$ and $y_2 = y_{2(x)}$ with QP $\omega = c$ and QPC $\lambda > 0$, then its general solution is

$$y(x) = y_1(x) + \frac{1}{z(x)} = \frac{1}{1-C\phi(x)} y_1(x) + \left(1 - \frac{1}{1-C\phi(x)}\right) y_2(x) =$$

$$= \mu(x,C) y_1(x) + (1-\mu(x,C)) y_2(x),$$

i.e. if the solution is in the form $y(x) = y_1(x) + \dfrac{1}{C_1 a(x) + b(x)}$, then

$$a(x) = \frac{-1}{\varphi(x)(y_1(x)-y_2(x))},\ b(x) = \frac{-1}{y_1(x)-y_2(x)}.$$

Example 2.2. Let

$$y' + \frac{2\cos^2 x - 2\cos x - 1}{\sin x - \cos x} y + e^x y^2 + e^{-x}(\sin x \cos x + \frac{1}{\sin x - \cos x}) = 0.$$

This equation has particular solutions $y_1 = e^{-x}\sin x$ and $y_2(x) = e^{-x}\cos x$ that are QPF with QP $\omega = 2\pi$ and QPC $\lambda = e^{-2\pi}$. Since the coefficients $f(x)$, $g(x)$, $h(x)$ are QPF with the same QP $\omega = 2\pi$ and QPC 1, $e^{2\pi} = \dfrac{1}{\lambda}$, $e^{-2\pi} = \lambda$ respectively, according to the Theorem 2.3. every solution is QPF.

Indeed, the general solution is $y = e^{-x}\left(\sin x + \frac{e^{\sin x+\cos x}(\cos x - \sin x)}{C_1 + e^{\sin x+\cos x}}\right)$, that is QPF with QP $\omega = 2\pi$ and QPC $\lambda = e^{-2\pi}$.

Theorem 2.4. If

1^o DE (1) has QPS $y_1(x)$, $y_2(x)$ and $y_3(x)$ with QP $\omega = c$ and QPC $\lambda > 0$,

2^o the coefficients $f(x), g(x), h(x)$ in DE (1) are QPF with QP $\omega = c$ and QPC $\lambda_1 = 1, \lambda_2 = \frac{1}{\lambda}, \lambda_3 = \lambda$ respectively,

3^o for $\psi(x) = \frac{y_3(x) - y_1(x)}{y_3(x) - y_2(x)}$, $z(x) = \frac{k\psi(x) - 1}{k\psi(x)} \frac{1}{y_2(x) - y_1(x)}$ is a general QPS with QP $\omega = c$ and QPC $\frac{1}{\lambda}$ to the equation (8),

then every solution for DE (1) is QPF with QP $\omega = c$ and QPC λ.

Proof. It can be proved in a similar manner as the Theorem 2.2.

Remark 2.3 The general solution for DE (1) is

$$y(x) = y_1(x) + \frac{1}{z(x)} = \frac{1}{1 - k\psi(x)} y_1(x) + \left(1 - \frac{1}{1 - k\psi(x)}\right) y_2(x) =$$

$$= \nu(x,k) y_1(x) + (1 - \nu(x,k)) y_2(x)$$

i.e. if $y(x) = y_1(x) + \frac{1}{C_2 a(x) + b(x)}$, then $a(x) = \frac{-1}{\psi(x)(y_1(x) - y_2(x))}$,

$b(x) = \frac{-1}{y_1(x) - y_2(x)}$.

Example 2.3. Let

$$y' + \frac{1-\sin x+\cos x+2\sin^2 x-2\sin^3 x}{(1-\sin x)(1-\cos x)(\cos x-\sin x)}y + e^x \frac{1-\sin x-\cos x}{(1-\sin x)(1-\cos x)(\cos x-\sin x)}y^2 +$$

$$+ e^{-x}\frac{-1-\sin x-\sin x\cos x-\sin^2 x\cos x+\sin^3 x}{(1-\sin x)(1-\cos x)(\cos x-\sin x)}y = 0\,.$$

This equation has particular solutions $y_1 = e^{-x}\sin x$, $y_2(x) = e^{-x}\cos x$ and $y_3(x) = e^{-x}$. They are QPF with QP $\omega = 2\pi$ and QPC $\lambda = e^{-2\pi}$. Since the coefficients $f(x)$, $g(x)$, $h(x)$ are QPF with the same QP $\omega = 2\pi$ and QPC 1, $e^{2\pi} = \frac{1}{\lambda}$, $e^{-2\pi} = \lambda$ respectively, according to the Theorem 2.4. every solution is QPF. Indeed, the general solution is

$$y = e^{-x}\left(\sin x + \frac{\cos x-\sin x-\sin x\cos x+\sin^2 x}{C(1-\cos x)+(1-\sin x)}\right),$$

that is QPF with QP $\omega = 2\pi$ and QPC $\lambda = e^{-2\pi}$.

Theorem 2.5. Let the DE (1) have one QPS with QP $\omega = c$ and QPC $\lambda = \frac{\lambda_3}{\lambda_1}$, and the coefficients $f(x) \neq 0, g(x), h(x)$ in DE (1) be QPF with QP $\omega = c$ and QPC $\lambda_1 \neq 1$, $\lambda_2 = \frac{1}{\lambda}, \lambda_3$, respectively. Then the QPS for DE (1) is

$$y = -\frac{h(x)}{f(x)} \tag{9}$$

if the relation

$$\left(\frac{h(x)}{f(x)}\right)' - g(x)\left(\frac{h(x)}{f(x)}\right)^2 = 0 \tag{10}$$

is satisfied.

Proof. Under the conditions, $f(x+c)=\lambda_1 f(x)$, $g(x+c)=\frac{1}{\lambda}g(x)$, $h(x+c)=\lambda_3 h(x)$, QPS to the DE (1) is also QPS to the equation

$$(\lambda_1-1)f(x)y=-\frac{1}{\lambda}(\lambda_3-\lambda)h(x),$$

from which we get

$$y=\mu_1\cdot\frac{h(x)}{f(x)} \tag{11}$$

where $\mu_1=-\frac{\lambda_3-\lambda}{\lambda(\lambda_1-1)}$. Solution (11) is QPF with QP $\omega=c$ and QPC $\lambda=\frac{\lambda_3}{\lambda_1}$ for which $\mu_1=-1$. Thus, from (11) we obtain (9). So, since the solution (9) is also the solution to DE (1), we obtain that the coefficients f,g,h have to satisfy the relation (10). □

Corollary 2.1. Under the conditions of the Theorem 2.5. QPS for DE (1) is given by

$$y=\frac{1}{\int_{x_0}^{x}g(x)dx+C(x_0)},C(x_0)=-G(x_0),G'(x)=g(x). \tag{12}$$

Proof. From the relation (10) we have

$$\frac{h(x)}{f(x)}=-\frac{1}{\int_{x_0}^{x}g(x)dx+C(x_0)}. \tag{13}$$

Since, under he conditions of the Theorem 2.5., QPS for DE (1) is $y=-\frac{h(x)}{f(x)}$, we obtain (12). □

Example 2.4. The Riccati equation

$$y' - e^{-x+\sin x}\cos x \cdot y + e^{-2x+\sin x}(-2+\cos x)\cdot y^2 + e^x \cos x = 0$$

has coefficients $f(x) = -e^{-x+\sin x}\cdot\cos x$, $g(x) = e^{-2x+\sin x}\cdot(-2+\cos x)$, $h(x) = e^x \cdot \cos x$, which are QPF with the same QP $\omega = 2\pi$ and QPC $\lambda_1 = e^{-2\pi}$, $\lambda_2 = e^{-4\pi}$, $\lambda_3 = e^{2\pi}$, respectively, and satisfy the condition (10). Thus, according to the Theorem 2.5, QPS for the given DE is $y = -\dfrac{h(x)}{f(x)} = e^{2x-\sin x}$ ($\omega = 2\pi$, $\lambda = e^{4\pi} = \dfrac{\lambda_3}{\lambda_1}$), or using (12):

$$y = -\frac{1}{\int\limits_{x_0}^{x} g(x)dx + C_0} = -\frac{1}{\int\limits_{x_0}^{x} e^{-2x+\sin x}\left(-2+\cos x\right)dx + C_0} = e^{2x-\sin x}.$$

Theorem 2.6. Let the coefficients $f(x), g(x), h(x)$ in DE (1) be QPF with QP $\omega = c$ and QPC $\lambda_1 \neq 1, \lambda_2 \neq \dfrac{1}{\lambda}, \lambda_3$ respectively.

1) If DE (1) has QPS $y = y(x)$ with QP $\omega = c$, QPC $\lambda \neq \dfrac{\lambda_1}{\lambda_2}$ and $f(x) \neq 0$, then $y = 0$.

2) If DE (1) has QPS $y = y(x)$ with QP $\omega = c$, QPC $\lambda \neq \dfrac{\lambda_1}{\lambda_2}$ and $f(x) = 0$, then $y = 0$.

3) If DE (1) has QPS $y = y(x)$ with QP $\omega = c$, QPC $\lambda = \dfrac{\lambda_1}{\lambda_2}$ and $h(x) = 0$, then $y = 0$ or $y = C$ when $Cg(x) + f(x) = 0$,

4) If DE (1) has QPS $y = y(x)$ with QP $\omega = c$, QPC $\lambda = \dfrac{\lambda_1}{\lambda_2}, \lambda_1 = \lambda_3\lambda_2$ i.e. $\lambda = \lambda_3$ and $Cg(x) + f(x) = 0$, then $y = C$.

5) If DE (1) has QPS $y = y(x)$ with QP $\omega = c$, QPC $\lambda = \dfrac{\lambda_1}{\lambda_2}$, $\lambda_1{}^2 = \lambda_2\lambda_3$, i.e. $\lambda^2 = \dfrac{\lambda_3}{\lambda_2}$ and $C^2 g(x) + Cf(x) + h(x) = 0$ then $y = C$.

Proof. Under the conditions of the theorem QPS for DE (1) and eq. (7) is also QPS to the equation

$$g(x)y^2 + pf(x)y + qh(x) = 0\,. \tag{14}$$

where

$$p = \frac{\lambda_1 - 1}{\lambda\lambda_2 - 1}, \quad q = \frac{1}{\lambda}\cdot\frac{\lambda_3 - \lambda}{\lambda\lambda_2 - 1}\,. \tag{15}$$

The last equation has solutions

$$y = \frac{-pf(x) \pm \sqrt{p^2 f^2(x) - 4qh(x)g(x)}}{2g(x)}\,, \tag{16}$$

Since QPS to DE (1) is QPS for eq. (14), it satisfies the equation

$$p\lambda(\lambda_1 - \lambda\lambda_2)f(x)y + q(\lambda_3 - \lambda^2\lambda_2)h(x) = 0\,. \tag{17}$$

Thus we have:

1) If $\lambda \neq \dfrac{\lambda_1}{\lambda_2}$, $\lambda_1^2 - \lambda_2\lambda_3 \neq 0$ and $f(x) \neq 0$ then,

$$y = \frac{(\lambda^2\lambda_2 - \lambda_3)q}{\lambda(\lambda_1 - \lambda\lambda_2)p}\cdot\frac{h(x)}{f(x)} = \mu_2\,\frac{h(x)}{f(x)},$$

from where follows $\lambda = \dfrac{\lambda_3}{\lambda_1}, p = q, \mu_2 = -1$ and $y = -\dfrac{h(x)}{f(x)}$.

If we compare the obtained solution with the solution (16) or substituting it in the eq. (14) we have $h(x) = 0$ and $y = 0$.

2) If $\lambda \neq \frac{\lambda_1}{\lambda_2}$, $\lambda_1^2 - \lambda_2\lambda_3 \neq 0$ and $f(x) = 0$ then we have the equation $g(x)y^2 + qh(x) = 0$ and $q(\lambda_3 - \lambda^2\lambda_2)h(x) = 0$ from where $h(x) = 0$ and $y = 0$.

3) If $\lambda = \frac{\lambda_1}{\lambda_2}$, (i.e. $p = 1$) and $h(x) = 0$ then QPS is $y = 0$ or $y = C$ when $Cg(x) + f(x) = 0$.

4) If $\lambda = \frac{\lambda_1}{\lambda_2}$ and $\lambda = \lambda_3$, i.e. $\lambda_1 = \lambda_2\lambda_3$, from where follows $p = 1$ and $q = 0$, then QPS is $y = 0$ or $y = C$ if $Cg(x) + f(x) = 0$.

5) If $\lambda = \frac{\lambda_1}{\lambda_2}$ and $\lambda^2 = \frac{\lambda_3}{\lambda_2}$, i.e. $\lambda_1^2 = \lambda_2\lambda_3$, then $p = 1, q = 1$ and QPS is $y = 0$ or $y = C$ if $C^2 g(x) + Cf(x) + h(x) = 0$.

Remark 2.4. From eq. (14), after short transformations, we obtain that QPS for DE (1) is also a solution to the equation

$$y' + (1 - p)f(x)y + (1 - q)h(x) = 0 \tag{18}$$

where p and q are gien by (15).

Under the conditions of the Theorem 2.6., solving eq. (18) as a linear DE, we can find all QPS for DE (1). They are given by the formula

$$y = e^{-\alpha\gamma(x)}\left(C - \delta(x)\right), \tag{19}$$

where $p = 1 - \alpha$, $q = 1 - \beta$, $\gamma(x) = \int f(x)dx$, $\delta(x) = \beta\int e^{\alpha\gamma(x)}h(x)dx$. In this case, from here we have also then only $y(x) = 0$ or $y(x) = C$ are QPS for DE (1).

Remark 2.5. Some examples in [4] are special cases from the obtained results in this paper.

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Р е з и м е

КВАЗИ-ПЕРИОДИЧНО РЕШЕНИЕ НА ДИФЕРЕНЦИЈАЛНАТА РАВЕНКА НА РИКАТИ

Во овој труд даваме некои услови за постоење квазипериодични решенија на Рикатиевата диференцијална равенка и ги наоѓаме решенијата за случај на константен квази-период.

Клучни зборови: диференцијална равенка (DE); квази-период (QP); квазипериодично решение (QPS); квази-периодичен коефициент (QPC); равенка (eq.)

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