GENERALIZATION OF A SUMMATION DUE TO RAMANUJAN

Tibor K. Pogány, Arjun K. Rathie, Ujjawal Pandey

Abstract: The aim of this research note is to find the sum of the series
\[ 1 + \frac{x - 1}{x + 1 + j} + \frac{(x - 1)(x - 2)}{(x + 1 + j)(x + 2 + j)} + \ldots \ (\Re \{x\} > 0) \]
for \( j = 0, 1, 2, 3, 4, 5 \). When \( j = 0 \), we get a summation due to Ramanujan. The results are derived with the help of generalized Kummer’s theorem obtained already by Lavoie, Grandie and Rathie.

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Key words and Phrases: Hypergeometric \( _2F_1 \); Kummer’s summation theorem; Ramanujan summation formula

1. INTRODUCTION

We start with an interesting summation due to Ramanujan [4], viz.
\[ 1 + \frac{x - 1}{x + 1} + \frac{(x - 1)(x - 2)}{(x + 1)(x + 2)} + \ldots = \frac{2^{2x-1} \Gamma^2(x+1)}{\Gamma(2x+1)} \ (\Re \{x\} > 0). \] (1)
As pointed out by Berndt [2], this summation can be obtained quite simply by employing Kummer’s summation theorem [1] viz.
\[ _2F_1 \left[ a, b ; 1+a-b \right] = \frac{\sqrt{\pi} \Gamma(1+a-b)}{2^a \Gamma \left( 1 + \frac{a-1}{2} \right) \Gamma \left( \frac{a+1}{2} \right)} \] (2)
by taking \( a = 1 \) and \( b = 1 - x \).
In 1996 Lavoie, Grandie and Rathie [3] have obtained explicit expressions of
\[ {}_2F_1 \left[ \begin{array}{c} a, b \\ 1 + a - b + j \end{array}; -1 \right] \]
for \( j = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5 \). The case \( j = 0 \) was presented above in (2) as the Kummer's theorem. However, the following another special cases with non-negative \( j \) will be required in our present investigations:

\[ {}_2F_1 \left[ \begin{array}{c} a, b \\ 1 + a - b + j \end{array}; -1 \right] = \frac{\sqrt{\pi} \Gamma(1 + a - b + j) \Gamma(1 - b)}{2^a \Gamma \left( 1 - b + \frac{1}{2} \right)} \]

\[ \times \left[ \frac{A_j}{\Gamma \left( \frac{a}{2} - b + \frac{j}{2} + 1 \right) \Gamma \left( \frac{a}{2} + \frac{j}{2} + \frac{1}{2} - \left[ \frac{j}{2} \right] \right)} \right] + \frac{B_j}{\Gamma \left( \frac{a}{2} - b + \frac{j}{2} + \frac{1}{2} \right) \Gamma \left( \frac{a}{2} + \frac{1}{2} - \left[ \frac{j}{2} \right] \right)} \]

where, as usual, \( [x] \) denotes the greatest integer less then or equal to \( x \) and the values of the constants \( A_j, B_j \) are given in the Table 1. below.

<table>
<thead>
<tr>
<th>( j )</th>
<th>( A_j )</th>
<th>( B_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( a - b + 1 )</td>
<td>-2</td>
</tr>
<tr>
<td>3</td>
<td>( 3b - 2a - 5 )</td>
<td>( 2a - b + 1 )</td>
</tr>
<tr>
<td>4</td>
<td>( 2(a - b + 3)(1 + a - b) - (b - 1)(b - 4) )</td>
<td>-( 4(a - b + 2) )</td>
</tr>
<tr>
<td>5</td>
<td>( -4(6 + a - b)^2 + 2(b + 11) \times (6 + a - b) + b^2 - 13b - 20 )</td>
<td>( 4(6 + a - b)^2 + 2(b - 17) \times (6 + a - b) - b^2 - b + 62 )</td>
</tr>
</tbody>
</table>
The aim of this note is to find an interesting generalization of the Ramanujan’s summation (1) by making sense of (4). Five summations closely related to Ramanujan’s summation have also been obtained as special cases of our main findings. The summations derived here are simple, interesting easily established and may be useful.

2. MAIN SUMMATION

Theorem 1.

\[
1 + \frac{x-1}{x+1+j} + \frac{(x-1)(x-2)}{(x+1+j)(x+2+j)} + \ldots
\]

\[
= \sqrt{\pi}(1+j) \frac{\Gamma(x)}{2} \left[ \frac{A_j}{\Gamma\left(\frac{x+1+j}{2}\right)\Gamma\left(1+j\left[\frac{1+j}{2}\right]\right)} \right. \\
\left. + \frac{B_j}{\Gamma\left(x+\frac{j}{2}\right)\Gamma\left(\frac{1+j}{2} - \left[\frac{j}{2}\right]\right)} \right]
\]

for \( j = 0, 1, 2, 3, 4, 5 \). The coefficients \( A_j, B_j \) can be obtained from the table by changing \( a \) by 1 and \( b \) by \( 1-x \) respectively.

Proof. In (4) taking \( a = 1 \) and \( b = 1-x \), then expressing the hypergeometric function as a series, we have

\[
2F_1\left[\begin{array}{c} 1, 1-x \\ 1+x+j \end{array} \right] = -1 = \sum_{n=0}^{\infty} \frac{(1)(1-x)_n}{(1+x+j)_n} \frac{(-1)^n}{n!} \\
= 1 + \frac{x-1}{x+1+j} + \frac{(x-1)(x-2)}{(x+1+j)(x+2+j)} + \ldots
\]
Here \((a)_n := 1, (a)_n = a(a - 1) \ldots (a - n + 1), \ n \in \mathbb{N}\) stands for the Pochhammer-symbol, called sometimes shifted factorial as well.

Similarly, putting \(a = 1, \ b = 1 - x\) on the right–hand expression in (4), we get

\[
\frac{\Gamma \left( \frac{3}{2} \right) \Gamma (x) \Gamma (1 + x + j)}{\Gamma (x + j)} \left[ \frac{A_j}{\Gamma \left( x + \frac{1}{2} + \frac{j}{2} \right) \Gamma \left( 1 + \frac{j}{2} - \left[ \frac{1+j}{2} \right] \right)} \right]
\]

\[
+ \frac{B_j}{\Gamma \left( x + \frac{j}{2} \right) \Gamma \left( \frac{1}{2} + \frac{j}{2} - \left[ \frac{j}{2} \right] \right)}
\]

such that, after easy simplification, one transforms into the asserted right–hand expression of (5). This completes the derivation of (5).

\[ \square \]

3. SPECIAL CASES

In (5), if we take \(j = 0, 1, 2, 3, 4, 5\) we have the following interesting summations.

1. For \(j = 0\)

\[
1 + \frac{x - 1}{x + 1} \frac{(x - 1)(x - 2)}{(x + 1)(x + 2)} + \ldots = \frac{\Gamma \left( \frac{3}{2} \right) \Gamma (1 + x)}{\Gamma \left( x + \frac{1}{2} \right)}
\]

The right–hand side of (8) can be seen equivalent to the right–hand side of (5).
2. For $j = 1$

\[
1 + \frac{x-1}{x+6} + \frac{(x-1)(x-2)}{(x+6)(x+7)} + \ldots
\]

\[
= (1 + x) \Gamma(x) \Gamma \left( \frac{3}{2} \right) \left[ \frac{1}{\Gamma \left( x + \frac{1}{2} \right)} - \frac{1}{\Gamma \left( \frac{1}{2} \right) \Gamma(x+1)} \right]. \tag{9}
\]

3. For $j = 2$

\[
1 + \frac{x-1}{x+3} + \frac{(x-1)(x-2)}{(x+3)(x+4)} + \ldots
\]

\[
= (x + 2) \Gamma(x) \Gamma \left( \frac{3}{2} \right) \left[ \frac{1 + x}{\Gamma \left( x + \frac{3}{2} \right)} - \frac{2}{\Gamma \left( \frac{1}{2} \right) \Gamma(x+1)} \right]. \tag{10}
\]

4. For $j = 3$

\[
1 + \frac{x-1}{x+4} + \frac{(x-1)(x-2)}{(x+4)(x+5)} + \ldots
\]

\[
= (x + 3) \Gamma(x) \Gamma \left( \frac{3}{2} \right) \left[ \frac{x+2}{\Gamma \left( x + \frac{3}{2} \right)} - \frac{3x+4}{\Gamma \left( \frac{1}{2} \right) \Gamma(x+2)} \right]. \tag{11}
\]

5. For $j = 4$

\[
1 + \frac{x-1}{x+5} + \frac{(x-1)(x-2)}{(x+5)(x+6)} + \ldots
\]

\[
= (x + 2) (x + 4) \Gamma(x) \Gamma \left( \frac{3}{2} \right) \left[ \frac{x+3}{\Gamma \left( x + \frac{5}{2} \right)} - \frac{4}{\Gamma \left( \frac{1}{2} \right) \Gamma(x+2)} \right]. \tag{12}
\]
For $j = 5$

$$1 + \frac{x - 1}{x + 6} + \frac{(x - 1)(x - 2)}{(x + 6)(x + 7)} + \ldots$$

$$= (x + 5) \Gamma(x) \Gamma\left(\frac{3}{2}\right) \frac{x^2 + 7x + 12}{\Gamma\left(x + \frac{5}{2}\right)} - \frac{5x^2 + 25x + 32}{\Gamma\left(\frac{1}{2}\right) \Gamma(x + 3)}.$$  \hspace{1cm} (13)

Clearly, (8) is a Ramanujan's summation and other summations (9) to (13) are seen to be closely related (8).

**Remark 1.** For another summations due to Ramanujan and their generalizations the interested reader can consult [5].

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**REFERENCES**


Резиме

ГЕНЕРАЛИЗАЦИЈА НА СУМИРАЊЕТО НА RAMANUJAN

Целта на ова истражување е да се најде сумата на редот

\[ 1 + \frac{x - 1}{x + 1 + j} + \frac{(x - 1)(x - 2)}{(x + 1 + j)(x + 2 + j)} + \ldots \quad (\Re(x) > 0) \]

за \( j = 0, 1, 2, 3, 4, 5 \). За \( j = 0 \), ја добиваме сумата на Ramanujan. Резултатите се добили со помош на генерализираната теорема на Kummer, дадена од Lavoie, Grandie и Rathie.

Ключни зборови: хипергеометриска \( _2F_1 \); сумирање на теоремата на Kummer; сумациона формула на Ramanujan

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