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SYSTEMS OF DIFFERENCE EQUATIONS APPROXIMATING THE LORENZ SYSTEM OF DIFFERENTIAL EQUATIONS

Biljana Zlatanovska, Dončo Dimovski

A b s t r a c t: In this paper, starting from the Lorenz system of differential equations, some systems of difference equations are produced. Using some regularities in these systems of difference equations, polynomial approximations of their solutions are found. Taking these approximations as coefficients, three power series are obtained and by computer calculations is examined that these power series are local approximations of the solutions of the starting Lorentz system of differential equations.

Key words: Lorenz system; differential equations; difference equations; power series; approximation

1. INTRODUCTION

The use of power series is one of the oldest methods for examining differential equations. In the literature there are numerous papers concerned with such a use of power series, like the papers [1], [2] and [3].

We consider the well known Lorenz system of differential equations:

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= x(r-z)-y \\ \dot{z} &= xy-bz \end{aligned} \qquad (1.1)$$

with three parameters *σ, r, b*.

For initial values $a_0 = x(0),\ b_0 = y(0),\ c_0 = z(0)$, we assume that the solutions of the Lorenz system are expanded as Maclaurin series, (1.2).

$$x(t)=a_0+a_1t+a_2\frac{t^2}{2!}+\ldots+a_n\frac{t^n}{n!}+\ldots=\sum_{n=0}^{\infty}a_n\frac{t^n}{n!}$$
$$y(t)=b_0+b_1t+b_2\frac{t^2}{2!}+\ldots+b_n\frac{t^n}{n!}+\ldots=\sum_{n=0}^{\infty}b_n\frac{t^n}{n!} \qquad (1.2)$$
$$z(t)=c_0+c_1t+c_2\frac{t^2}{2!}+\ldots+c_n\frac{t^n}{n!}+\ldots=\sum_{n=0}^{\infty}c_n\frac{t^n}{n!}$$

Using consecutive differentiation of (1.1) and the representation (1.2), for every $n \in N$, we obtain the following system of difference equations.

$$\begin{aligned} a_n &= \sigma(b_{n-1}-a_{n-1}) \\ b_n &= ra_{n-1}-b_{n-1}-\sum_{i=0}^{n-1}\binom{n-1}{i}a_i c_{n-i-1} \\ c_n &= -bc_{n-1}+\sum_{i=0}^{n-1}\binom{n-1}{i}a_i b_{n-i-1} \end{aligned} \qquad (1.3)$$

Our aim is to express the coefficients a_n, b_n and c_n, as polynomials in variables σ, r, b, a_0, b_0 and c_0. Separately, for each of the coefficients a_n, b_n and c_n we transform the system (1.3), by introducing new variables.

2. THE COEFFICIENT a_n

We write the system (1.3) in the form (2.1).

$$\begin{aligned} a_n &= \sigma(b_{n-1}-a_{n-1}) \\ b_n &= ra_{n-1}-b_{n-1}-a_0c_{n-1}-\sum_{i=1}^{n-1}\binom{n-1}{i}a_i c_{n-1-i} \\ c_n &= a_0b_{n-1}-bc_{n-1}+\sum_{i=1}^{n-1}\binom{n-1}{i}a_i b_{n-1-i} \end{aligned} \qquad (2.1)$$

For a fixed $n \in N$ and for any $1 \le k \le n$ we represent the coefficients a_n as:

$$a_n=\varphi_k^n a_{n-k}+\psi_k^n b_{n-k}+\xi_k^n c_{n-k}+\sum_{m=0}^{k-1}\sum_{i=k-m}^{n-m-1}\tau_k^n(i,m)a_i b_{n-i-m-1} +\sum_{m=0}^{k-1}\sum_{i=k-m}^{n-m-1}\pi_k^n(i,m)a_i c_{n-i-m-1},$$

where $\varphi_k^n, \psi_k^n, \xi_k^n, \tau_k^n(i.m), \pi_k^n(i,m)$ are new variables. For the new variables we obtain the new systems of difference equations:

$$\begin{aligned}
&\varphi_k^n = -\sigma\varphi_{k-1}^n + r\psi_{k-1}^n \\
&\psi_k^n = \sigma\varphi_{k-1}^n - \psi_{k-1}^n + a_0\xi_{k-1}^n + \sum_{i=1}^{k-1} a_i \binom{n-k+i}{i} \xi_{k-1-i}^n \\
&\xi_k^n = -a_0\psi_{k-1}^n - b\xi_{k-1}^n - \sum_{i=1}^{k-1} a_i \binom{n-k+i}{i} \psi_{k-1-i}^n \\
&\tau_k^n(i,m) = \xi_m^n \binom{n-m-1}{i}, \qquad \pi_k^n(i,m) = -\psi_m^n \binom{n-m-1}{i}
\end{aligned} \tag{2.2}$$

with initial values $\varphi_0^n = 1, \psi_0^n = 0, \xi_0^n = 0$, and transform the presentation of a_n to

$$a_n = \varphi_k^n a_{n-k} + \psi_k^n b_{n-k} + \xi_k^n c_{n-k} + \sum_{m=0}^{k-1} \sum_{i=k-m}^{n-m-1} \binom{n-1-m}{i} \xi_m^n a_i b_{n-i-m-1}$$
$$- \sum_{m=0}^{k-1} \sum_{i=k-m}^{n-m-1} \binom{n-1-m}{i} \psi_m^n a_i c_{n-i-m-1} .$$

For $n = k$, directly from the presentation of a_n, the inequality $n-m-1 < n-m$, implies:

2.1. $a_n = \varphi_n^n a_0 + \psi_n^n b_0 + \xi_n^n c_0$

Next, for fixed $n,k \in N$, for all $1 \le q \le k$ we take the following presentations:

$$\varphi_k^n = \alpha_q^{n,k} \varphi_{k-q}^n + \beta_q^{n,k} \psi_{k-q}^n + \gamma_q^{n,k} \xi_{k-q}^n + \sum_{m=0}^{q-1} \sum_{i=q-m}^{k-m-1} \iota_q^{n,k}(i,m)\, a_i \tag{2.3}$$

$$\psi_k^n = \overline{\alpha}_q^{n,k} \varphi_{k-q}^n + \overline{\beta}_q^{n,k} \psi_{k-q}^n + \overline{\gamma}_q^{n,k} \xi_{k-q}^n + \sum_{m=0}^{q-1} \sum_{i=q-m}^{k-m-1} \overline{\iota}_q^{n,k}(i,m)\, a_i \tag{2.4}$$

$$\xi_k^n = \overline{\overline{\alpha}}_q^{n,k} \varphi_{k-q}^n + \overline{\overline{\beta}}_q^{n,k} \psi_{k-q}^n + \overline{\overline{\gamma}}_q^{n,k} \xi_{k-q}^n + \sum_{m=0}^{q-1} \sum_{i=q-m}^{k-m-1} \overline{\overline{\iota}}_q^{n,k}(i,m)\, a_i \ . \tag{2.5}$$

For the presentation (2.3) we obtain the system of difference equations (2.6).

$$\begin{aligned}
\alpha_q^{n,k} &= -\sigma\,\alpha_{q-1}^{n,k} + \sigma\,\beta_{q-1}^{n,k} \\
\beta_q^{n,k} &= r\alpha_{q-1}^{n,k} - \beta_{q-1}^{n,k} - a_0\gamma_{q-1}^{n,k} - \sum_{i=1}^{q-1}\gamma_{q-1-i}^{n,k}\binom{n-k+q-1}{i}a_i \\
\gamma_q^{n,k} &= a_0\beta_{q-1}^{n,k} - b\gamma_{q-1}^{n,k} + \sum_{i=1}^{q-1}\beta_{q-1-i}^{n,k}\binom{n-k+q-1}{i}a_i \\
\iota_q^{n,k}(i,m) &= \binom{n-k+i+m}{i}\left[\beta_m^{n,k}\xi_{k-m-i-1}^{n} - \gamma_m^{n,k}\psi_{k-m-i-1}^{n}\right]
\end{aligned} \tag{2.6}$$

with the initial values $\alpha_0^{n,k}=1,\ \beta_0^{n,k}=0,\ \gamma_0^{n,k}=0,\ \iota_0^{n,k}=0$.

For the presentation (2.4) we obtain the system of difference equations:

$$\begin{aligned}
\overline{\alpha}_q^{n,k} &= -\sigma\,\overline{\alpha}_{q-1}^{n,k} + \sigma\,\overline{\beta}_{q-1}^{n,k} \\
\overline{\beta}_q^{n,k} &= r\overline{\alpha}_{q-1}^{n,k} - \overline{\beta}_{q-1}^{n,k} - a_0\overline{\gamma}_{q-1}^{n,k} - \sum_{i=1}^{q-1}\overline{\gamma}_{q-1-i}^{n,k}\binom{n-k+q-1}{i}a_i \\
\overline{\gamma}_q^{n,k} &= a_0\overline{\beta}_{q-1}^{n,k} - b\overline{\gamma}_{q-1}^{n,k} + \sum_{i=1}^{q-1}\overline{\beta}_{q-1-i}^{n,k}\binom{n-k+q-1}{i}a_i \\
\overline{\iota}_q^{n,k}(i,m) &= \binom{n-k+i+m}{i}\left[\overline{\beta}_m^{n,k}\xi_{k-m-i-1}^{n} - \overline{\gamma}_m^{n,k}\psi_{k-m-i-1}^{n}\right]
\end{aligned} \tag{2.7}$$

with the initial values $\overline{\alpha}_0^{n,k}=0,\ \overline{\beta}_0^{n,k}=1,\ \overline{\gamma}_0^{n,k}=0,\ \overline{\iota}_0^{n,k}=0$.

For the presentation (2.5) we obtain the system of difference equations:

$$\begin{aligned}
\overline{\overline{\alpha}}_q^{n,k} &= -\sigma\overline{\overline{\alpha}}_{q-1}^{n,k} + \sigma\overline{\overline{\beta}}_{q-1}^{n,k} \\
\overline{\overline{\beta}}_q^{n,k} &= r\overline{\overline{\alpha}}_{q-1}^{n,k} - \overline{\overline{\beta}}_{q-1}^{n,k} - a_0\overline{\overline{\gamma}}_{q-1}^{n,k} - \sum_{i=1}^{q-1}\overline{\overline{\gamma}}_{q-1-i}^{n,k}\binom{n-k+q-1}{i}a_i \\
\overline{\overline{\gamma}}_q^{n,k} &= a_0\overline{\overline{\beta}}_{q-1}^{n,k} - b\overline{\overline{\gamma}}_{q-1}^{n,k} + \sum_{i=1}^{q-1}\overline{\overline{\beta}}_{q-1-i}^{n,k}\binom{n-k+q-1}{i}a_i \\
\overline{\overline{\iota}}_q^{n,k}(i,m) &= \binom{n-k+i+m}{i}\left[\overline{\overline{\beta}}_m^{n,k}\xi_{k-m-i-1}^{n} - \overline{\overline{\gamma}}_m^{n,k}\psi_{k-m-i-1}^{n}\right]
\end{aligned} \tag{2.8}$$

with the initial values $\overline{\overline{\alpha}}_0^{n,k}=0,\ \overline{\overline{\beta}}_0^{n,k}=0,\ \overline{\overline{\gamma}}_0^{n,k}=1,\ \overline{\overline{\iota}}_0^{n,k}=0$.

For $k=q$, directly from the presentations (2.3), (2.4) and (2.5), the inequality $k-m-1<k-m$, together with $\varphi_0^n=1,\psi_0^n=0,\xi_0^n=0$, implies:

2.2. $\varphi_k^n=\alpha_k^{n,k}$ **,** $\psi_k^n=\overline{\alpha}_k^{n,k}$ and $\xi_k^n=\overline{\overline{\alpha}}_k^{n,k}$ **.**

For $n=k=q$, **2.1.** and **2.2.** imply:

2.3. $a_n=\alpha_n^{n,n}a_0+\overline{\alpha}_n^{n,n}b_0+\overline{\overline{\alpha}}_n^{n,n}c_0$ **.**

With all this, the question of finding a solution for a_n is transformed to the question of solving the systems of difference equations (2.6), (2.7) and (2.8).

We start with the system (2.6). By finding expressions for $\alpha_q^{n,k}$, $\beta_q^{n,k}$ and $\gamma_q^{n,k}$ for several small values of *n, k, q* we found that there are some regularities, and after long calculations, we obtained the following presentations:

$$\alpha_q^{n,k}=X_q^{n,k}+L_q^{n,k}\qquad \beta_q^{n,k}=Y_q^{n,k}+M_q^{n,k}\qquad \gamma_q^{n,k}=N_q^{n,k}, \tag{2.9}$$

where:

$$X_q^{n,k}=(-1)^{q-1}\sigma\left\{-(\sigma+r)^{q-1}+\sum_{m=1}^{\left[\frac{q-1}{2}\right]}\sum_{j=m+1}^{q-m}\binom{q-j}{m}\binom{j-1}{m-1}\sigma^{q-j-1}\left(r^j-r^m\right)\right\} \tag{2.10}$$

$$Y_q^{n,k}=(-1)^{q-1}\left\{(\sigma+r)^{q-1}r-\sum_{m=1}^{\left[\frac{q}{2}\right]}\sum_{j=m+1}^{q-m+1}\binom{q-j}{m-1}\binom{j-1}{m-1}\sigma^{q-j}\left(r^j-r^m\right)\right\} \tag{2.11}$$

and $L_q^{n,k}, M_q^{n,k}, N_q^{n,k}$ satisfy the following system of difference equations:

$$\begin{aligned} L_q^{n,k}&=-\sigma L_{q-1}^{n,k}+\sigma M_{q-1}^{n,k}\\ M_q^{n,k}&=rL_{q-1}^{n,k}-M_{q-1}^{n,k}-a_0N_{q-1}^{n,k}-\sum_{i=1}^{q-1}N_{q-1-i}^{n,k}\binom{n-k+q-1}{i}a_i\\ N_q^{n,k}&=a_0Y_{q-1}^{n,k}+a_0M_{q-1}^{n,k}-bN_{q-1}^{n,k}+\sum_{i=1}^{q-1}\left(Y_{q-1-i}^{n,k}+M_{q-1-i}^{n,k}\right)\binom{n-k+q-1}{i}a_i \end{aligned} \tag{2.12}$$

with $X_0^{n,k}=1,\ Y_0^{n,k}=0,\ L_0^{n,k}=0,\ M_0^{n,k}=0,\ N_0^{n,k}=0$.

By **2.2.** and **2.3.**, it is enough to find $\varphi_n^n=\alpha_n^{n,n}=X_n^{n,n}+L_n^{n,n}$, and since $X_n^{n,n}$ is known, we have to find $L_n^{n,n}$. Each $L_q^{n,k}, M_q^{n,k}, N_q^{n,k}$ is a polynomial with variables σ, r, b, a_0, b_0 and c_0. For convenience with the signs, we use the notation: $\hat{L}_q^{n,k} = (-1)^{q-1} L_q^{n,k}$. By suitably grouping the parts in these polynomials and calculating the first several of them, we found that:

$$\hat{L}_1^{n,n} = \hat{L}_2^{n,n} = \hat{L}_3^{n,n} = 0; \qquad \hat{L}_4^{n,n} = \sigma\{ra_0^2\};$$

$$\hat{L}_5^{n,n} = \sigma\{2(ra_0^2) + 6\sigma(ra_0^2) + b(ra_0^2) - 5\sigma ra_0 b_0\};$$

$$\begin{aligned}\hat{L}_6^{n,n} = \sigma\{&3(ra_0^2) + 16\sigma(ra_0^2) + 2b(ra_0^2) + 32\sigma^2(ra_0^2) + 8\sigma b(ra_0^2) + b^2(ra_0^2)\\ &+ 11\sigma r(ra_0^2) - (ra_0^2)a_0^2 - 9\sigma c_0(ra_0^2) - 21\sigma ra_0b_0 - 37\sigma(\sigma ra_0 b_0)\\ &- 6b(\sigma ra_0 b_0) + 8\sigma^2 rb_0^2\};\end{aligned}$$

$$\begin{aligned}\hat{L}_7^{n,n} = \sigma\{&4(ra_0^2) + 27\sigma(ra_0^2) + 3b(ra_0^2) + 82\sigma^2(ra_0^2) + 18\sigma b(ra_0^2) + 2b^2(ra_0^2)\\ &+ 122\sigma^3(ra_0^2) + 40\sigma^2 b(ra_0^2) + 9\sigma b^2(ra_0^2) + b^3(ra_0^2) + 45\sigma r(ra_0^2)\\ &+ 108\sigma\sigma r(ra_0^2) + 15b\sigma r(ra_0^2) - 3(ra_0^2)a_0^2 - 16\sigma(ra_0^2)a_0^2 - 2b(ra_0^2)a_0^2\\ &- 39\sigma c_0(ra_0^2) - 74\sigma\sigma c_0(ra_0^2) - 27b\sigma c_0(ra_0^2) + 28\sigma a_0b_0(ra_0^2) - 60\sigma ra_0b_0\\ &- 173\sigma(\sigma ra_0b_0) - 27b(\sigma ra_0b_0) - 161\sigma^2(\sigma ra_0b_0) - 47\sigma b(\sigma ra_0b_0)\\ &- 7b^2(\sigma ra_0b_0) - 63\sigma r(\sigma ra_0b_0) + 49\sigma c_0(\sigma ra_0b_0) + 58(\sigma^2 rb_0^2)\\ &+ 58\sigma(\sigma^2 rb_0^2) + 10b(\sigma^2 rb_0^2)\};\end{aligned}$$

and for $q>7$, the polynomial $\hat{L}_q^{n,n}$ has the form of the polynomial

$$A\hat{L}_{q-1}^{n,n} + B\hat{L}_{q-2}^{n,n} + C\hat{L}_{q-3}^{n,n} + D\hat{L}_{q-4}^{n,n}$$

where $A = 1+\sigma+b$, $B = \sigma r - a_0^2 - \sigma c_0$, $C = \sigma a_0 b_0$, $D = -\sigma^2 b_0^2$. So, for $q>7$ we choose to approximate $\hat{L}_q^{n,n}$ with the solutions $\hat{L}_q^{n,n}(\approx)$ of the difference equation:

$$\hat{L}_q^{n,n}(\approx) = A\hat{L}_{q-1}^{n,n}(\approx) + B\hat{L}_{q-2}^{n,n}(\approx) + C\hat{L}_{q-3}^{n,n}(\approx) + D\hat{L}_{q-4}^{n,n}(\approx) \qquad (2.13)$$

with the initial values $\hat{L}_4^{n,n}, \hat{L}_5^{n,n}, \hat{L}_6^{n,n}\ \hat{L}_7^{n,n}$.

To solve this difference equation, we take the following representation

$$\hat{L}_q^{n,n}(\approx) = P_w^1\hat{L}_{q-w}^{n,n}(\approx) + P_w^2\hat{L}_{q-w-1}^{n,n}(\approx) + P_w^3\hat{L}_{q-w-2}^{n,n}(\approx) + P_w^4\hat{L}_{q-w-3}^{n,n}(\approx) \qquad (2.14)$$

where $1 \le w \le q$ and $P_0^1 = 1, P_0^2 = 0, P_0^3 = 0, P_0^4 = 0$, $P_1^1 = A, P_1^2 = B, P_1^3 = C, P_1^4 = D$.

Using the presentation (2.14) we obtain the following system of difference equations:

$$\begin{array}{ll} P_w^1 = A\,P_{w-1}^1 + P_{w-1}^2 & P_w^2 = B\,P_{w-1}^1 + P_{w-1}^3 \\ P_w^3 = C\,P_{w-1}^1 + P_{w-1}^4 & P_w^4 = D\,P_{w-1}^1 \end{array} . \tag{2.15}$$

The solutions of the system (2.15) are the polynomials:

$$P_w^1 = A^w + \sum_{j=1}^{\left[\frac{w}{2}\right]} \binom{w-j}{j} A^{w-2j} B^j + \sum_{j=1}^{\left[\frac{w}{3}\right]} \binom{w-2j}{j} A^{w-3j} C^j + \sum_{j=1}^{\left[\frac{w}{4}\right]} \binom{w-3j}{j} A^{w-4j} D^j$$

$$+ \sum_{m=1}^{\left[\frac{w-2}{3}\right]} \sum_{j=m+1}^{\left[\frac{w-m}{2}\right]} \binom{j}{m} \binom{w-j-m}{j} A^{w-2j-m} B^{j-m} C^m$$

$$+ \sum_{m=1}^{\left[\frac{w-2}{4}\right]} \sum_{j=m+1}^{\left[\frac{w-2m}{2}\right]} \binom{j}{m} \binom{w-j-2m}{j} A^{w-2j-2m} B^{j-m} D^m$$

$$+ \sum_{m=1}^{\left[\frac{w-3}{4}\right]} \sum_{j=m+1}^{\left[\frac{w-m}{3}\right]} \binom{j}{m} \binom{w-2j-m}{j} A^{w-3j-m} C^{j-m} D^m +$$

$$+ \sum_{s=1}^{\left[\frac{w-5}{4}\right]} \sum_{m=s+1}^{\left[\frac{w-s-2}{3}\right]} \sum_{j=m}^{\left[\frac{w-m-s-2}{2}\right]} \binom{m}{s} \binom{j+1}{m} \binom{w-j-m-s-1}{j+1} A^{w-2j-m-s-2} B^{j-m+1} C^{m-s} D^s$$

$$P_w^2 = \sum_{j=1}^{\left[\frac{w+1}{2}\right]} \binom{w-j}{j-1} A^{w-2j+1} B^j + \sum_{j=1}^{\left[\frac{w+1}{3}\right]} \binom{w-2j}{j-1} A^{w-3j+1} C^j + \sum_{j=1}^{\left[\frac{w+1}{4}\right]} \binom{w-3j}{j-1} A^{w-4j+1} D^j$$

$$+ \sum_{m=1}^{\left[\frac{w-1}{3}\right]} \sum_{j=m+1}^{\left[\frac{w-m+1}{2}\right]} \binom{j}{m} \binom{w-j-m}{j-1} A^{w-2j-m+1} B^{j-m} C^m$$

$$+ \sum_{m=1}^{\left[\frac{w-1}{4}\right]} \sum_{j=m+1}^{\left[\frac{w-2m+1}{2}\right]} \binom{j}{m} \binom{w-j-2m}{j-1} A^{w-2j-2m+1} B^{j-m} D^m$$

$$+ \sum_{m=1}^{\left[\frac{w-2}{4}\right]} \sum_{j=m+1}^{\left[\frac{w-m+1}{3}\right]} \binom{j}{m} \binom{w-2j-m}{j-1} A^{w-3j-m+1} C^{j-m} D^m$$

$$+ \sum_{s=1}^{\left[\frac{w-4}{4}\right]} \sum_{m=s+1}^{\left[\frac{w-s-1}{3}\right]} \sum_{j=m}^{\left[\frac{w-m-s-1}{2}\right]} \binom{m}{s} \binom{j+1}{m} \binom{w-j-m-s-1}{j} A^{w-2j-m-s-1} B^{j-m+1} C^{m-s} D^s$$

$$P_w^3 = \sum_{j=1}^{\left[\frac{w+2}{3}\right]} \binom{w-2j+1}{j-1} A^{w-3j+2} C^j + \sum_{j=1}^{\left[\frac{w+2}{4}\right]} \binom{w-3j+1}{j-1} A^{w-4j+2} D^j$$
$$+ \sum_{m=1}^{\left[\frac{w}{3}\right]} \sum_{j=m}^{\left[\frac{w-m}{2}\right]} \binom{j}{m-1} \binom{w-j-m}{j} A^{w-2j-m} B^{j-m+1} C^m$$
$$+ \sum_{m=1}^{\left[\frac{w}{4}\right]} \sum_{j=m}^{\left[\frac{w-2m}{2}\right]} \binom{j}{m-1} \binom{w-j-2m}{j} A^{w-2j-2m} B^{j-m+1} D^m$$
$$+ \sum_{m=1}^{\left[\frac{w-1}{4}\right]} \sum_{j=m}^{\left[\frac{w-m-1}{3}\right]} \binom{j+1}{m} \binom{w-2j-m-1}{j} A^{w-3j-m-1} C^{j-m+1} D^m$$
$$+ \sum_{s=1}^{\left[\frac{w-3}{4}\right]} \sum_{m=s+1}^{\left[\frac{w-s}{3}\right]} \sum_{j=m}^{\left[\frac{w-m-s}{2}\right]} \binom{m}{s} \binom{j}{m-1} \binom{w-j-m-s}{j} A^{w-2j-m-s} B^{j-m+1} C^{m-s} D^s$$

and

$$P_w^4 = \sum_{j=1}^{\left[\frac{w+3}{4}\right]} \binom{w-3j+2}{j-1} A^{w-4j+3} D^j$$
$$+ \sum_{m=1}^{\left[\frac{w+1}{4}\right]} \sum_{j=m}^{\left[\frac{w-2m+1}{2}\right]} \binom{j}{m-1} \binom{w-j-2m+1}{j} A^{w-2j-2m+1} B^{j-m+1} D^m$$
$$+ \sum_{m=1}^{\left[\frac{w}{4}\right]} \sum_{j=m}^{\left[\frac{w-m}{3}\right]} \binom{j}{m-1} \binom{w-2j-m}{j} A^{w-3j-m} C^{j-m+1} D^m .$$
$$+ \sum_{s=1}^{\left[\frac{w-2}{4}\right]} \sum_{m=s+1}^{\left[\frac{w-s+1}{3}\right]} \sum_{j=m}^{\left[\frac{w-m-s+1}{2}\right]} \binom{m-1}{s-1} \binom{j}{m-1} \binom{w-j-m-s+1}{j} A^{w-2j-m-s+1} B^{j-m+1} C^{m-s} D^s .$$

For $w = q - 7$:

$$\hat{L}_q^{n,n}(\approx) = P_{q-7}^1 \hat{L}_7^{n,n} + P_{q-7}^2 \hat{L}_6^{n,n} + P_{q-7}^3 \hat{L}_5^{n,n} + P_{q-7}^4 \hat{L}_4^{n,n},$$

and from **2.2.**, for $n = q$, we obtain:

$$\varphi_n^n(\approx) = \alpha_n^{n,n}(\approx) = X_n^{n,n} + L_n^{n,n}(\approx)$$

$$= (-1)^{n-1}\sigma\left\{-(\sigma+r)^{n-1} + \sum_{m=1}^{\left[\frac{n-1}{2}\right]}\sum_{j=m+1}^{n-m}\binom{n-j}{m}\binom{j-1}{m-1}\sigma^{n-j-1}\left(r^j - r^m\right)\right\}$$

$$+ (-1)^{n-1}\left\{P_{n-7}^1\,\hat{L}_7^{n,n} + P_{n-7}^2\,\hat{L}_6^{n,n} + P_{n-7}^3\,\hat{L}_5^{n,n} + P_{n-7}^4\,\hat{L}_4^{n,n}\right\}.$$

Next, we continue with the system (2.7). Similarly as above we set:

$$\overline{\alpha}_q^{n,k} = \overline{X}_q^{n,k} + \overline{L}_q^{n,k}, \qquad \overline{\beta}_q^{n,k} = \overline{Y}_q^{n,k} + \overline{M}_q^{n,k}, \qquad \overline{\gamma}_q^{n,k} = \overline{N}_q^{n,k}$$

where:

$$\overline{X}_q^{n,k} = (-1)^{q-1}\sigma\left\{(\sigma+r)^{q-1} - \sum_{m=1}^{\left[\frac{q}{2}\right]}\sum_{j=m}^{q-m}\binom{q-1-j}{m-1}\binom{j}{m-1}\sigma^{q-j-1}\left(r^j - r^{m-1}\right)\right\}$$

$$\overline{Y}_q^{n,k} = (-1)^{q-1}\left\{-(\sigma+r)^{q-1}r + \left(r^q - 1\right) + \sum_{m=2}^{\left[\frac{q+1}{2}\right]}\sum_{j=m}^{q-m+1}\binom{q-1-j}{m-2}\binom{j}{m-1}\sigma^{q-j}\left(r^j - r^{m-1}\right)\right\}$$

and $\overline{L}_q^{n,k}, \overline{M}_q^{n,k}, \overline{N}_q^{n,k}$ satisfy the difference equations (2.16),

$$\begin{aligned}
\overline{L}_q^{n,k} &= -\sigma\overline{L}_{q-1}^{n,k} + \sigma\overline{M}_{q-1}^{n,k}\\
\overline{M}_q^{n,k} &= r\overline{L}_{q-1}^{n,k} - \overline{M}_{q-1}^{n,k} - a_0\,\overline{N}_{q-1}^{n,k} - \sum_{i=1}^{q-1}\overline{N}_{q-1-i}^{n,k}\binom{n-k+q-1}{i}a_i\\
\overline{N}_q^{n,k} &= a_0\,\overline{Y}_{q-1}^{n,k} + a_0\,\overline{M}_{q-1}^{n,k} - b\overline{N}_{q-1}^{n,k} + \sum_{i=1}^{q-1}\left(\overline{Y}_{q-1-i}^{n,k} + \overline{M}_{q-1-i}^{n,k}\right)\binom{n-k+q-1}{i}a_i
\end{aligned} \tag{2.16}$$

with $\overline{X}_0^{n,k} = 0, \overline{Y}_0^{n,k} = 1,\ \overline{L}_0^{n,k} = 0,\ \overline{M}_0^{n,k} = 0,\ \overline{N}_0^{n,k} = 0$. The system (2.16) is the system (2.12), but with different notations, and different initial values.

By **2.2.** and **2.3.**, it is enough to find $\overline{\varphi}_n^n = \overline{\alpha}_n^{n,n} = \overline{X}_n^{n,n} + \overline{L}_n^{n,n}$, and since $\overline{X}_n^{n,n}$ is known, we have to find $\overline{L}_n^{n,n}$. Using the notation: $\hat{\overline{L}}_q^{n,k} = (-1)^{q-1}\,\overline{L}_q^{n,k}$, we calculated $\hat{\overline{L}}_q^{n,k}$ for, q from 1 to 6:

$$\hat{\overline{L}}_1^{n,n} = \hat{\overline{L}}_2^{n,n} = 0; \quad \hat{\overline{L}}_3^{n,n} = \sigma(-a_0^2); \quad \hat{\overline{L}}_4^{n,n} = \sigma\left\{-2\left(a_0^2\right) - 4\sigma\left(a_0^2\right) - b\left(a_0^2\right) + 3\sigma a_0 b_0\right\};$$

$$\hat{L}_5^{n,n} = \sigma\{-3(a_0^2) - 10\sigma(a_0^2) - 2b(a_0^2) - 11\sigma^2(a_0^2) - 5\sigma b(a_0^2) - b^2(a_0^2) - 6\sigma r(a_0^2) + (a_0^2)a_0^2 + 4\sigma c_0(a_0^2) + 12\sigma a_0 b_0 + 13\sigma(\sigma a_0 b_0) + 4b(\sigma a_0 b_0) - 3\sigma^2 b_0^2\};$$

$$\hat{L}_6^{n,n} = \sigma\{-4(a_0^2) - 18\sigma(a_0^2) - 3b(a_0^2) - 34\sigma^2(a_0^2) - 12\sigma b(a_0^2) - 2b^2(a_0^2) - 26\sigma^3(a_0^2) - 16\sigma^2 b(a_0^2) - 6\sigma b^2(a_0^2) - b^3(a_0^2) - 24\sigma r(a_0^2) - 38\sigma\sigma r(a_0^2) - 9b\sigma r(a_0^2) + 3(a_0^2)a_0^2 + 11\sigma(a_0^2)a_0^2 + 2b(a_0^2)a_0^2 + 18\sigma c_0(a_0^2) + 24\sigma\sigma c_0(a_0^2) + 12b\sigma c_0(a_0^2) - 15\sigma a_0 b_0(a_0^2) + 33\sigma a_0 b_0 + 62\sigma(\sigma a_0 b_0) + 17b(\sigma a_0 b_0) + 38\sigma^2(\sigma a_0 b_0) + 19\sigma b(\sigma a_0 b_0) + 5b^2(\sigma a_0 b_0) + 25\sigma r(\sigma a_0 b_0) - 15\sigma c_0(\sigma a_0 b_0) - 21(\sigma^2 b_0^2) - 13\sigma(\sigma^2 b_0^2) - 4b(\sigma^2 b_0^2)\}.$$

By the same argument as above, we arrive to the same difference equations as (2.13) and (2.14), but for q>6, and to the same equation (2.15). With all this we obtain:

$$\psi_n^n(\approx) = \overline{\alpha}_n^{n,n}(\approx) = \overline{X}_n^{n,n} + \overline{L}_n^{n,n}(\approx)$$
$$= (-1)^{n-1}\sigma\left\{(\sigma + r)^{n-1} - \sum_{m=1}^{\left[\frac{n}{2}\right]}\sum_{j=m}^{n-m}\binom{n-1-j}{m-1}\binom{j}{m-1}\sigma^{n-j-1}\left(r^j - r^{m-1}\right)\right\}$$
$$+ (-1)^{n-1}\left\{P_{n-6}^1\,\hat{L}_6^{n,n} + P_{n-6}^2\,\hat{L}_5^{n,n} + P_{n-6}^3\,\hat{L}_4^{n,n} + P_{n-6}^4\,\hat{L}_3^{n,n}\right\}.$$

For the system (2.8), we set:

$$\overline{\overline{\alpha}}_q^{n,k} = \overline{\overline{L}}_q^{n,k}, \qquad \overline{\overline{\beta}}_q^{n,k} = \overline{\overline{M}}_q^{n,k}, \qquad \overline{\overline{\gamma}}_q^{n,k} = (-b)^q + \overline{\overline{N}}_q^{n,k}$$

where $\overline{\overline{L}}_q^{n,k}, \overline{\overline{M}}_q^{n,k}, \overline{\overline{N}}_q^{n,k}$ satisfy the difference equations (2.17),

$$\begin{aligned}
\overline{\overline{L}}_q^{n,k} &= -\sigma\overline{\overline{L}}_{q-1}^{n,k} + \sigma\overline{\overline{M}}_{q-1}^{n,k} \\
\overline{\overline{M}}_q^{n,k} &= r\overline{\overline{L}}_{q-1}^{n,k} - \overline{\overline{M}}_{q-1}^{n,k} - a_0\overline{\overline{N}}_{q-1}^{n,k} - \sum_{i=1}^{q-1}\overline{\overline{N}}_{q-1-i}^{n,k}\binom{n-k+q-1}{i}a_i \qquad (2.17) \\
\overline{\overline{N}}_q^{n,k} &= a_0\overline{\overline{M}}_{q-1}^{n,k} - b\overline{\overline{N}}_{q-1}^{n,k} + \sum_{i=1}^{q-1}\overline{\overline{M}}_{q-1-i}^{n,k}\binom{n-k+q-1}{i}a_i
\end{aligned}$$

with $\overline{\overline{L}}_0^{n,k} = 0,\ \overline{\overline{M}}_0^{n,k} = 0,\ \overline{\overline{N}}_0^{n,k} = 0$. The system (2.17) is analogous to the system (2.12) (i.e. the system (2.16)), but with different notations, and different initial values.

By **2.2.** and **2.3.**, it is enough to find $\overline{\overline{\varphi}}_n^{\,n} = \overline{\overline{\alpha}}_n^{\,n,n} = \overline{\overline{L}}_n^{\,n,n}$. Using the notation: $\hat{\overline{\overline{L}}}_q^{n,k} = (-1)^{q-1}\overline{\overline{L}}_q^{\,n,k}$, we calculated $\hat{\overline{\overline{L}}}_q^{n,k}$ for q from 1 to 5:

$$\hat{\overline{\overline{L}}}_1^{n,n} = 0; \qquad \hat{\overline{\overline{L}}}_2^{n,n} = \sigma\, a_0; \qquad \hat{\overline{\overline{L}}}_3^{n,n} = \sigma\{(a_0) + 2\sigma(a_0) + b(a_0) - \sigma b_0\};$$

$$\begin{aligned}\hat{\overline{\overline{L}}}_4^{n,n} = {}&\sigma\{(a_0) + 2\sigma(a_0) + b(a_0) + 3\sigma^2(a_0) + 3\sigma b(a_0) + b^2(a_0) + 2\sigma r(a_0)\\ &-(a_0)a_0^2 - \sigma c_0(a_0) - 2\sigma b_0 - 2\sigma(\sigma b_0) - 2b(\sigma b_0)\};\end{aligned}$$

$$\begin{aligned}\hat{\overline{\overline{L}}}_5^{n,n} = {}&\sigma\{(a_0) + 2\sigma(a_0) + b(a_0) + 3\sigma^2(a_0) + 3\sigma b(a_0) + b^2(a_0) + 4\sigma^3(a_0)\\ &+ 6\sigma^2 b(a_0) + 4\sigma b^2(a_0) + b^3(a_0) + 4\sigma r(a_0) + 6\sigma\sigma r(a_0) + 4b\sigma r(a_0)\\ &- 2(a_0)a_0^2 - 7\sigma(a_0)a_0^2 - 2b(a_0)a_0^2 - 2\sigma c_0(a_0) - 3\sigma\sigma c_0(a_0) - 4b\sigma c_0(a_0)\\ &+ 7\sigma a_0 b_0(a_0) - 3\sigma b_0 - 4\sigma(\sigma b_0) - 5b(\sigma b_0) - 3\sigma^2(\sigma b_0) - 5\sigma b(\sigma b_0)\\ &- 3b^2(\sigma b_0) - 2\sigma r(\sigma b_0) + \sigma(\sigma b_0)c_0\}.\end{aligned}$$

Again, by the same argument as above, we arrive to the same difference equations as (2.13) and (2.14), but for q>5, and to the same equation (2.15). With all this, we have

$$\xi_n^n(\approx) = \overline{\overline{\alpha}}_n^{\,n,n}(\approx) = \overline{\overline{L}}_n^{\,n,n}(\approx) = (-1)^{n-1}\left\{P_{n-5}^1\,\hat{\overline{\overline{L}}}_5^{n,k} + P_{n-5}^2\,\hat{\overline{\overline{L}}}_4^{n,n} + P_{n-5}^3\,\hat{\overline{\overline{L}}}_3^{n,n} + P_{n-5}^4\,\hat{\overline{\overline{L}}}_2^{n,n}\right\}.$$

The above discussion and **2.3.**, produce the following approximation

$$a_n(\approx) \;= \alpha_n^{n,n}(\approx)\,a_0 + \overline{\alpha}_n^{\,n,n}(\approx)\,b_0 + \overline{\overline{\alpha}}_n^{\,n,n}(\approx)\,c_0$$

for a_n, n>7.

For $j \in \{1,2,3,4,5,6,7\}$ we calculate from (1.3) the exact values of a_j and we set $a_j(\approx) = a_j$ for $j \in \{0,1,2,3,4,5,6,7\}$.

3. THE COEFFICIENT c_n

For the coefficients c_n we consider the system of difference equations (2.1).

For an $n \in N$ and for any $1 \le k \le n$ we represent the coefficients c_n as:

$$c_n = \overline{\overline{\varphi}}_k^n a_{n-k} + \overline{\overline{\psi}}_k^n b_{n-k} + \overline{\overline{\xi}}_k^n c_{n-k} + \sum_{m=0}^{k-1} \sum_{i=k-m}^{n-m-1} \overline{\overline{\tau}}_k^n (i,m) a_i b_{n-i-m-1}$$
$$+ \sum_{m=0}^{k-1} \sum_{i=k-m}^{n-m-1} \overline{\overline{\pi}}_k^n (i,m) a_i c_{n-i-m-1}$$

where $\overline{\overline{\varphi}}_k^n$, $\overline{\overline{\psi}}_k^n$, $\overline{\overline{\xi}}_k^n$, $\overline{\overline{\tau}}_k^n (i.m), \overline{\overline{\pi}}_k^n (i,m)$ are new variables. For the new variables we obtain the new systems of difference equations:

$$\begin{aligned}
\overline{\overline{\varphi}}_k^n &= -\sigma \overline{\overline{\varphi}}_{k-1}^n + r \overline{\overline{\psi}}_{k-1}^n \\
\overline{\overline{\psi}}_k^n &= \sigma \overline{\overline{\varphi}}_{k-1}^n - \overline{\overline{\psi}}_{k-1}^n + a_0 \overline{\overline{\xi}}_{k-1}^n + \sum_{i=1}^{k-1} a_i \binom{n-k+i}{i} \overline{\overline{\xi}}_{k-1-i}^n \\
\overline{\overline{\xi}}_k^n &= -a_0 \overline{\overline{\psi}}_{k-1}^n - b \overline{\overline{\xi}}_{k-1}^n - \sum_{i=1}^{k-1} a_i \binom{n-k+i}{i} \overline{\overline{\psi}}_{k-1-i}^n \\
\overline{\overline{\tau}}_k^n (i,m) &= \overline{\overline{\xi}}_m^n \binom{n-m-1}{i}, \qquad \overline{\overline{\pi}}_k^n (i,m) = -\overline{\overline{\psi}}_m^n \binom{n-m-1}{i}
\end{aligned} \tag{3.1}$$

with $\overline{\overline{\varphi}}_0^n = 0, \overline{\overline{\psi}}_0^n = 0, \overline{\overline{\xi}}_0^n = 1$, and transform the presentation of c_n to:

$$c_n = \overline{\overline{\varphi}}_k^n a_{n-k} + \overline{\overline{\psi}}_k^n b_{n-k} + \overline{\overline{\xi}}_k^n c_{n-k} + \sum_{m=0}^{k-1} \sum_{i=k-m}^{n-m-1} \binom{n-1-m}{i} \overline{\overline{\xi}}_m^n a_i b_{n-i-m-1}$$
$$- \sum_{m=0}^{k-1} \sum_{i=k-tm}^{n-m-1} \binom{n-1-m}{i} \overline{\overline{\psi}}_m^n a_i c_{n-i-m-1}.$$

For $n = k$, the inequality $n - m - 1 < n - m$, implies:

3.1. $c_n = \overline{\overline{\varphi}}_n^n a_0 + \overline{\overline{\psi}}_n^n b_0 + \overline{\overline{\xi}}_n^n c_0$ **.**

The systems (3.1) are the same as the systems (2.2), with different letters and different initial values. We use the following presentations as for the coefficient a_n, with different symbols for all $1 \le q \le k$, at fixed $n, k \in N$:

$$\begin{aligned}
\overline{\overline{\varphi}}_k^n &= \rho_q^{n,k} \overline{\overline{\varphi}}_{k-q}^n + \mu_q^{n,k} \overline{\overline{\psi}}_{k-q}^n + \delta_q^{n,k} \overline{\overline{\xi}}_{k-q}^n + \sum_{m=0}^{q-1} \sum_{i=q-m}^{k-m-1} \Delta_q^{n,k} (i,m) a_i \\
\overline{\overline{\psi}}_k^n &= \overline{\rho}_q^{n,k} \overline{\overline{\varphi}}_{k-q}^n + \overline{\mu}_q^{n,k} \overline{\overline{\psi}}_{k-q}^n + \overline{\delta}_q^{n,k} \overline{\overline{\xi}}_{k-q}^n + \sum_{m=0}^{q-1} \sum_{i=q-m}^{k-m-1} \overline{\Delta}_q^{n,k} (i,m) a_i \\
\overline{\overline{\xi}}_k^n &= \overline{\overline{\rho}}_q^{n,k} \overline{\overline{\varphi}}_{k-q}^n + \overline{\overline{\mu}}_q^{n,k} \overline{\overline{\psi}}_{k-q}^n + \overline{\overline{\delta}}_q^{n,k} \overline{\overline{\xi}}_{k-q}^n + \sum_{m=0}^{q-1} \sum_{i=q-m}^{k-m-1} \overline{\overline{\Delta}}_q^{n,k} (i,m) a_i
\end{aligned}$$

$$\rho_q^{n,k} = U_q^{n,k} + S_q^{n,k} \qquad \bar{\rho}_q^{n,k} = \bar{U}_q^{n,k} + \bar{S}_q^{n,k} \qquad \bar{\bar{\rho}}_q^{n,k} = \bar{\bar{S}}_q^{n,k}$$

$$\mu_q^{n,k} = V_q^{n,k} + K_q^{n,k} \qquad \bar{\mu}_q^{n,k} = \bar{V}_q^{n,k} + \bar{K}_q^{n,k} \qquad \bar{\bar{\mu}}_q^{n,k} = \bar{\bar{K}}_q^{n,k}$$

$$\delta_q^{n,k} = T_q^{n,k} \qquad \bar{\delta}_q^{n,k} = \bar{T}_q^{n,k} \qquad \bar{\bar{\delta}}_q^{n,k} = (-b)^q + \bar{\bar{T}}_q^{n,k} \; ,$$

with the initial values:

$$\rho_0^{n,k} = \bar{\mu}_0^{n,k} = \bar{\bar{\delta}}_0^{n,k} = 1 \, , \bar{\rho}_0^{n,k} = \bar{\bar{\rho}}_0^{n,k} = \mu_0^{n,k} = \bar{\bar{\mu}}_0^{n,k} = \delta_0^{n,k} = \bar{\delta}_0^{n,k} = 0 \, ,$$

$$U_0^{n,k} = \bar{V}_0^{n,k} = 1, \bar{U}_0^{n,k} = V_0^{n,k} = S_0^{n,k} = K_0^{n,k} = \bar{S}_0^{n,k} = \bar{K}_0^{n,k} = \bar{\bar{S}}_0^{n,k} = \bar{\bar{K}}_0^{n,k} = 0, ,$$

$$T_0^{n,k} = \bar{T}_0^{n,k} = \bar{\bar{T}}_0^{n,k} = \Delta_0^{n,k} = \bar{\Delta}_0^{n,k} = \bar{\bar{\Delta}}_0^{n,k} = 0 \, .$$

For $q = k = n$, $\bar{\bar{\varphi}}_n^{\,n} = \delta_n^{n,n}$, $\bar{\bar{\psi}}_n^{\,n} = \bar{\delta}_n^{n,n}$, $\bar{\bar{\xi}}_n^{\,n} = \bar{\bar{\delta}}_n^{n,n}$, and by analogous discussion as for the coefficient a_n, we obtain the following approximation for the coefficient c_n, $n>5$,

$$\begin{aligned} c_n(\approx) &= \delta_n^{n,n} a_0 + \bar{\delta}_n^{n,n}(\approx) b_0 + \bar{\bar{\delta}}_n^{n,n}(\approx) c_0 \\ &= (-1)^{n-1}\{b^n c_0\} + (-1)^{n-1}\{P_{n-4}^1 \hat{\bar{T}}_4^{n,n} + P_{n-4}^2 \hat{\bar{T}}_3^{n,n} + P_{n-4}^3 \hat{\bar{T}}_2^{n,n} + P_{n-4}^4 \hat{\bar{T}}_1^{n,n}\} b_0 \\ &\quad + (-1)^{n-1}\{P_{n-5}^1 \left(\hat{T}_5^{n,n} a_0 + \hat{\bar{\bar{T}}}_5^{n,n} c_0 \right) + P_{n-5}^2 \left(\hat{T}_4^{n,n} a_0 + \hat{\bar{\bar{T}}}_4^{n,n} c_0 \right) \\ &\quad + P_{n-5}^3 \left(\hat{T}_3^{n,n} a_0 + \hat{\bar{\bar{T}}}_3^{n,n} c_0 \right) + P_{n-5}^4 \left(\hat{T}_2^{n,n} a_0 + \hat{\bar{\bar{T}}}_2^{n,n} c_0 \right) \} , \end{aligned}$$

where the initial values for $\hat{T}_j^{n,n}, \hat{\bar{T}}_j^{n,n}, \hat{\bar{\bar{T}}}_j^{n,n}$, are:

$$\hat{T}_1^{n,n} = 0; \qquad \hat{T}_2^{n,n} = a_0 r; \qquad \hat{T}_3^{n,n} = -a_0 r - 3\sigma a_0 r - b a_0 r + 2\sigma r b_0 ;$$

$$\begin{aligned} \hat{T}_4^{n,n} = \{ &-(a_0 r) - 4\sigma(a_0 r) - b(a_0 r) - 7\sigma^2(a_0 r) - 3\sigma b(a_0 r) - b^2(a_0 r) \\ &- 4\sigma(a_0 r) r + (a_0 r) a_0^2 + 3\sigma(a_0 r) c_0 + 6(\sigma r b_0) + 6\sigma(\sigma r b_0) + 2b(\sigma r b_0) \}; \end{aligned}$$

$$\begin{aligned} \hat{T}_5^{n,n} = \{ &-(a_0 r) - 5\sigma(a_0 r) - b(a_0 r) - 11\sigma^2(a_0 r) - 4\sigma b(a_0 r) - b^2(a_0 r) - 15\sigma^3(a_0 r) \\ &- 7\sigma^2 b(a_0 r) - 3\sigma b^2(a_0 r) - b^3(a_0 r) - 12\sigma(a_0 r) r - 20\sigma\sigma(a_0 r) r - 4b\sigma(a_0 r) r \\ &+ 2(a_0 r) a_0^2 + 10\sigma(a_0 r) a_0^2 + 2b(a_0 r) a_0^2 + 10\sigma(a_0 r) c_0 + 14\sigma\sigma(a_0 r) c_0 \\ &+ 7b\sigma(a_0 r) c_0 - 13\sigma(a_0 r) a_0 b_0 + 14(\sigma r b_0) + 20\sigma(\sigma r b_0) + 6b(\sigma r b_0) \\ &+ 14\sigma^2(\sigma r b_0) + 6\sigma b(\sigma r b_0) + 2b^2(\sigma r b_0) + 8\sigma(\sigma r b_0) r - 4\sigma(\sigma r b_0) c_0 \} ; \end{aligned}$$

$$\hat{\bar{T}}_1^{n,n} = a_0 ; \qquad \hat{\bar{T}}_2^{n,n} = a_0 + \sigma a_0 + b a_0 - \sigma b_0 ;$$

$$\hat{\bar{T}}_3^{n,n} = \{a_0 + 2\sigma a_0 + b a_0 + \sigma^2 a_0 + \sigma b a_0 + b^2 a_0 + 2\sigma r a_0 - a_0 a_0^2 - \sigma c_0 a_0 - 3(\sigma b_0) - \sigma(\sigma b_0) - b(\sigma b_0)\};$$

$$\hat{\bar{T}}_4^{n,n} = \{a_0 + 3\sigma a_0 + b a_0 + 3\sigma^2 a_0 + 2\sigma b a_0 + b^2 a_0 + \sigma^3 a_0 + \sigma^2 b a_0 + \sigma b^2 a_0 + b^3 a_0 + 6\sigma a_0 r + 6\sigma\sigma r a_0 + 2\sigma b r a_0 - 2a_0 a_0^2 - 6\sigma a_0 a_0^2 - 2b a_0 a_0^2 - 4\sigma a_0 c_0 - 2\sigma\sigma a_0 c_0 - 2\sigma b a_0 c_0 + 7 a_0 \sigma a_0 b_0 - 7\sigma b_0 - 4\sigma(\sigma b_0) - 3b(\sigma b_0) - \sigma^2(\sigma b_0) - \sigma b(\sigma b_0) - b^2(\sigma b_0) - 4\sigma r(\sigma b_0) + \sigma c_0(\sigma b_0)\};$$

$$\hat{\bar{\bar{T}}}_1^{n,n} = 0; \qquad \hat{\bar{\bar{T}}}_2^{n,n} = a_0^2; \qquad \hat{\bar{\bar{T}}}_3^{n,n} = a_0^2 + 3\sigma a_0^2 + 2b a_0^2 - 3\sigma a_0 b_0;$$

$$\hat{\bar{\bar{T}}}_4^{n,n} = \{a_0^2 + 4\sigma a_0^2 + 2b a_0^2 + 7\sigma^2 a_0^2 + 8\sigma b a_0^2 + 3b^2 a_0^2 + 5\sigma r a_0^2 - a_0^2 a_0^2 - 4\sigma c_0 a_0^2 - 8\sigma a_0 b_0 - 10\sigma(\sigma a_0 b_0) - 8b(\sigma a_0 b_0) + 3\sigma^2 b_0^2\};$$

$$\hat{\bar{\bar{T}}}_5^{n,n} = \{a_0^2 + 5\sigma a_0^2 + 2b a_0^2 + 11\sigma^2 a_0^2 + 10\sigma b a_0^2 + 3b^2 a_0^2 + 15\sigma^3 a_0^2 + 24\sigma^2 b a_0^2 + 15\sigma b^2 a_0^2 + 4b^3 a_0^2 + 14\sigma r a_0^2 + 26\sigma\sigma r a_0^2 + 15 b\sigma r a_0^2 - 2a_0^2 a_0^2 - 10\sigma a_0^2 a_0^2 - 3b a_0^2 a_0^2 - 12\sigma c_0 a_0^2 - 20\sigma\sigma c_0 a_0^2 - 18 b\sigma c_0 a_0^2 + 15\sigma a_0 b_0 a_0^2 - 17\sigma a_0 b_0 - 30\sigma(\sigma a_0 b_0) - 23b(\sigma a_0 b_0) - 25\sigma^2(\sigma a_0 b_0) - 35\sigma b(\sigma a_0 b_0) - 15b^2(\sigma a_0 b_0) - 20\sigma r(\sigma a_0 b_0) + 15\sigma(\sigma a_0 b_0)c_0 + 14\sigma^2 b_0^2 + 10\sigma(\sigma^2 b_0^2) + 11b(\sigma^2 b_0^2)\}.$$

For $j \in \{1,2,3,4,5\}$ we calculate from (1.3) the exact values of c_j and we set $c_j(\approx) = c_j$ for $j \in \{0,1,2,3,4,5\}$.

4. THE COEFFICIENT b_n

For the coefficient b_n we write the system (1.3) in the form:

$$a_n = \sigma(b_{n-1} - a_{n-1})$$

$$b_n = (r - c_0)a_{n-1} - b_{n-1} - \sum_{i=0}^{n-2}\binom{n-1}{i} a_i\, c_{n-1-i}$$

$$c_n = b_0\, a_{n-1} - b\, c_{n-1} + \sum_{i=0}^{n-2}\binom{n-1}{i} a_i\, b_{n-1-i}$$

and for a fixed $n \in N$ and any $1 \le k \le n$ we represent b_n as:

$$b_n = \overline{\varphi}_k^n a_{n-k} + \overline{\psi}_k^n b_{n-k} + \overline{\xi}_k^n c_{n-k} + \sum_{m=0}^{k-1} \sum_{i=k-m}^{n-k-1} \overline{\tau}_k^n (i,m) a_i b_{n-i-m-1}$$
$$+ \sum_{m=0}^{k-1} \sum_{i=k-m}^{n-k-1} \overline{\pi}_k^n (i,m) a_i c_{n-i-m-1}$$

where $\overline{\varphi}_k^n, \overline{\psi}_k^n, \overline{\xi}_k^n$, $\overline{\tau}_k^n(i.m), \overline{\pi}_k^n(i,m)$ are new variables. For the new variables we obtain the new systems of difference equations:

$$\begin{aligned} &\overline{\varphi}_k^n = -\sigma \overline{\varphi}_{k-1}^n + (r - c_0)\overline{\psi}_{k-1}^n - \sum_{i=1}^{k-1} \binom{n-k+i}{n-k} c_i \overline{\psi}_{k-1-i}^n \\ &\overline{\psi}_k^n = \sigma \overline{\varphi}_{k-1}^n - \overline{\psi}_{k-1}^n \qquad \overline{\xi}_k^n = 0 \\ &\overline{\tau}_k^n(i,m) = 0 \qquad \overline{\pi}_k^n(i,m) = -\overline{\psi}_k^n \binom{n-m-1}{i}. \end{aligned} \tag{4.1}$$

with $\overline{\varphi}_0^n = 0, \overline{\psi}_0^n = 1, \overline{\xi}_0^n = 0$, and transform the presentation of b_n to:

$$b_n = \overline{\varphi}_k^n a_{n-k} + \overline{\psi}_k^n b_{n-k} - \sum_{m=0}^{k-1} \sum_{i=0}^{n-k-1} \binom{n-1-m}{i} \overline{\psi}_m^n a_i c_{n-i-m-1} .$$

For $n = k$, the presentation of b_n and the inequality $n - k - 1 < 0$, imply:

4.1. $b_n = \overline{\varphi}_n^n a_0 + \overline{\psi}_n^n b_0$.

Similarly as (2.3) and (2.4), for fixed $n, k \in N$ and any $1 \le q \le k$, we represent $\overline{\varphi}_k^n, \overline{\psi}_k^n$ as:

$$\overline{\varphi}_k^n = \eta_q^{n,k} \overline{\varphi}_{k-q}^n + \omega_q^{n,k} \overline{\psi}_{k-q}^n + \sum_{m=0}^{q-1} \sum_{i=q-m}^{k-m-1} \kappa_q^{n,k}(i,m) c_i \tag{4.2}$$

$$\overline{\psi}_k^n = \overline{\eta}_q^{n,k} \overline{\varphi}_{k-q}^n + \overline{\omega}_q^{n,k} \overline{\psi}_{k-q}^n + \sum_{m=0}^{q-1} \sum_{i=q-m}^{k-m-1} \overline{\kappa}_q^{n,k}(i,m) c_i \quad . \tag{4.3}$$

Again, similarly as (2,6) and (2.7) we obtain the following systems:

$$\begin{aligned} &\eta_q^{n,k} = -\sigma \eta_{q-1}^{n,k} + \sigma \omega_{q-1}^{n,k} \\ &\omega_q^{n,k} = (r - c_0)\eta_{q-1}^{n,k} - \omega_{q-1}^{n,k} - \sum_{i=1}^{q-1} \eta_{q-1-i}^{n,k} \binom{n-k+q-1}{i} c_i \\ &\kappa_q^{n,k}(i,m) = -\binom{n-k+i+m}{i} \eta_m^{n,k} \overline{\psi}_{k-m-i-1}^n \end{aligned} \tag{4.4}$$

$$\begin{aligned}
\overline{\eta}_q^{\,n,k} &= -\sigma\, \overline{\eta}_{q-1}^{\,n,k} + \sigma\, \overline{\omega}_{q-1}^{\,n,k} \\
\overline{\omega}_q^{\,n,k} &= \left(r - c_0\right)\overline{\eta}_{q-1}^{\,n,k} - \overline{\omega}_{q-1}^{\,n,k} - \sum_{i=1}^{q-1} \overline{\eta}_{q-1-i}^{\,n,k} \binom{n-k+q-1}{i} c_i \\
\overline{\kappa}_q^{\,n,k}(i,m) &= -\binom{n-k+i+m}{i} \overline{\eta}_m^{\,n,k}\, \overline{\psi}_{k-m-i-1}^{\,n}
\end{aligned} \tag{4.5}$$

with initial values $\eta_0^{n,k} = 1, \omega_0^{n,k} = 0,\ \overline{\eta}_0^{\,n,k} = 0, \overline{\omega}_0^{\,n,k} = 1$.

For $q = k = n$, $\overline{\varphi}_n^{\,n} = \omega_n^{n,k}, \overline{\psi}_n^{\,n} = \overline{\omega}_n^{\,n,n}$, and by **4.1.** we obtain:

4.2. $b_n = \omega_n^{n,n} a_0 + \overline{\omega}_n^{\,n,n} b_0$.

By the same discussion as for the coefficient a_n we arrived to the following presentations:

$$\omega_q^{n,k} = W_q^{n,k} + E_q^{n,k} \qquad \overline{\omega}_q^{\,n,k} = \overline{W}_q^{\,n,k} + \overline{E}_q^{\,n,k}$$

where:

$$W_q^{n,k} = (-1)^{q-1}\left[\sigma + (r-c_0)\right]^{q-1}(r-c_0) + (-1)^{q-1}\sum_{m=1}^{\left[\frac{q}{2}\right]}\sum_{j=m+1}^{q-m+1}\binom{q-j}{m-1}\binom{j-1}{m-1}\sigma^{q-j}\left[(r-c_0)^j - (r-c_0)^m\right],$$

$$\overline{W}_q^{\,n,k} = (-1)^q\left[\sigma + (r-c_0)\right]^{q-1}(r-c_0) + (-1)^{q-1}\left[(r-c_0)^q - 1\right] + (-1)^{q-1}\sum_{m=2}^{\left[\frac{q+1}{2}\right]}\sum_{j=m}^{q-m+1}\binom{q-j-1}{m-2}\binom{j}{m-1}\sigma^{q-j}\left[(r-c_0)^j - (r-c_0)^{m-1}\right],$$

and $\hat{E}_q^{n,k} = (-1)^{q-1} E_q^{n,k}$, $\hat{\overline{E}}_q^{\,n,k} = (-1)^{q-1}\overline{E}_q^{\,n,k}$, have the form of the polynomials:

$A\,\hat{E}_{q-1}^{n,k} + B\,\hat{E}_{q-2}^{n,k} + \overline{C}\,\hat{E}_{q-3}^{n,k} + D\,\hat{E}_{q-4}^{n,k}$ and $A\,\hat{\overline{E}}_{q-1}^{\,n,k} + B\,\hat{\overline{E}}_{q-2}^{\,n,k} + \overline{C}\,\hat{\overline{E}}_{q-3}^{\,n,k} + D\,\hat{\overline{E}}_{q-4}^{\,n,k}$,

for: $A = 1 + \sigma + b,\quad B = \sigma(r - c_0) - a_0^2,\quad \overline{C} = \sigma a_0 b_0 - \sigma b c_0,\quad D = -\sigma^2 b_0^2$.

So, for q>5 we approximate $\hat{E}_q^{n,n}$ with the solutions $\hat{E}_q^{n,n}(\approx)$ of the difference equation:

$$\hat{E}_q^{n,n}(\approx) = A\,\hat{E}_{q-1}^{n,n}(\approx) + B\,\hat{E}_{q-2}^{n,n}(\approx) + \overline{C}\,\hat{E}_{q-3}^{n,n}(\approx) + D\,\hat{E}_{q-4}^{n,n}(\approx), \tag{4.6}$$

with the initial values $\hat{E}_2^{n,n}$, $\hat{E}_3^{n,n}$, $\hat{E}_4^{n,n}$ $\hat{E}_5^{n,n}$.

In the same way as for a_n, we take the presentation

$$\hat{E}_q^{n,n}(\approx) = \overline{P}_w^1\,\hat{E}_{q-w}^{n,n}(\approx) + \overline{P}_w^2\,\hat{E}_{q-w-1}^{n,n}(\approx) + \overline{P}_w^3\,\hat{E}_{q-w-2}^{n,n}(\approx) + \overline{P}_w^4\,\hat{E}_{q-w-3}^{n,n}(\approx) \quad (4.7)$$

and obtain the system of difference equations

$$\begin{array}{ll} \overline{P}_w^1 = A\,\overline{P}_{w-1}^1 + \overline{P}_{w-1}^2 & \overline{P}_w^2 = B\,\overline{P}_{w-1}^1 + \overline{P}_{w-1}^3 \\ \overline{P}_w^3 = \overline{C}\,P_{w-1}^1 + \overline{P}_{w-1}^4 & \overline{P}_w^4 = D\,\overline{P}_{w-1}^1 \end{array}, \quad (4.8)$$

which is the same as the system (2.15) with C replaced by $\overline{C}$. So, the solutions of (4.8) are the polynomials $\overline{P}_w^1, \overline{P}_w^2, \overline{P}_w^3, \overline{P}_w^4$ obtained from the polynomials $P_w^1, P_w^2, P_w^3, P_w^4$ with C replaced by $\overline{C}$.

Similarly, for q>6 we approximate $\hat{\hat{E}}_q^{n,n}$ with the solutions $\hat{\hat{E}}_q^{n,n}(\approx)$ of the difference equation:

$$\hat{\hat{E}}_q^{n,n}(\approx) = A\,\hat{\hat{E}}_{q-1}^{n,n}(\approx) + B\,\hat{\hat{E}}_{q-2}^{n,n}(\approx) + \overline{C}\,\hat{\hat{E}}_{q-3}^{n,n}(\approx) + D\,\hat{\hat{E}}_{q-4}^{n,n}(\approx), \quad (4.9)$$

with the initial values $\hat{\hat{E}}_3^{n,n}$, $\hat{\hat{E}}_4^{n,n}$, $\hat{\hat{E}}_5^{n,n}$ $\hat{\hat{E}}_6^{n,n}$.

Similarly as for a_n, all this produces the following approximation:

$$\begin{aligned} b_n(\approx) = (-1)^{n-1}\left\{\overline{P}_{n-5}^1\,\hat{E}_5^{n,n} + \overline{P}_{n-5}^2\,\hat{E}_4^{n,n} + \overline{P}_{n-5}^3\,\hat{E}_3^{n,n} + \overline{P}_{n-5}^4\,\hat{E}_2^{n,n}\right\}a_0 + W_n^{n,n} \\ + (-1)^{n-1}\left\{\overline{P}_{n-6}^1\,\hat{\hat{E}}_6^{n,n} + \overline{P}_{n-6}^2\,\hat{\hat{E}}_5^{n,n} + \overline{P}_{n-6}^3\,\hat{\hat{E}}_4^{n,n} + \overline{P}_{n-6}^4\,\hat{\hat{E}}_3^{n,n}\right\}b_0 + \overline{W}_n^{n,n} \end{aligned}$$

where the initial values for $\hat{E}_j^{n,n}$ and $\hat{\hat{E}}_j^{n,n}$ are:

$$\hat{E}_1^{n,n} = 0; \qquad \hat{E}_2^{n,n} = -bc_0 + a_0 b_0;$$

$$\hat{E}_3^{n,n} = \left\{2(a_0b_0) + 3\sigma(a_0b_0) + b(a_0b_0) - (bc_0) - 2\sigma(bc_0) - b(bc_0) - a_0^2(r - c_0) - \sigma b_0^2\right\};$$

$$\hat{\hat{E}}_1^{n,n} = 0; \qquad \hat{\hat{E}}_2^{n,n} = 0; \qquad \hat{\hat{E}}_3^{n,n} = -2\sigma(a_0b_0) + 2\sigma(bc_0);$$

$$\begin{aligned} \hat{E}_4^{n,n} = \{&3(a_0b_0) + 8\sigma(a_0b_0) + 2b(a_0b_0) + 7\sigma^2(a_0b_0) + 4\sigma b(a_0b_0) + b^2(a_0b_0) \\ &+ 8\sigma(a_0b_0)(r - c_0) - (a_0b_0)a_0^2 - (bc_0) - 2\sigma(bc_0) - b(bc_0) - 3\sigma^2(bc_0) \\ &- 3\sigma b(bc_0) - b^2(bc_0) - 4\sigma(bc_0)(r - c_0) + (bc_0)a_0^2 - 2a_0^2(r - c_0) \\ &- 6\sigma a_0^2(r - c_0) - ba_0^2(r - c_0) - 4(\sigma b_0^2) - 4\sigma(\sigma b_0^2) - b(\sigma b_0^2)\}; \end{aligned}$$

$$\hat{\hat{E}}_4^{n,n} = \{-8(\sigma a_0 b_0) - 6\sigma(\sigma a_0 b_0) - 3b(\sigma a_0 b_0) + 5(\sigma b c_0) + 3\sigma(\sigma b c_0) + 3b(\sigma b c_0)$$
$$+ 3\sigma a_0^2 (r - c_0) + 3(\sigma^2 b_0^2)\};$$

$$\hat{E}_5^{n,n} = \{4(a_0 b_0) + 15\sigma(a_0 b_0) + 3b(a_0 b_0) + 24\sigma^2(a_0 b_0) + 10\sigma b(a_0 b_0) + 2b^2(a_0 b_0)$$
$$+ 15\sigma^3(a_0 b_0) + 11\sigma^2 b(a_0 b_0) + 5\sigma b^2(a_0 b_0) + b^3(a_0 b_0) + 32\sigma(a_0 b_0)(r - c_0)$$
$$+ 46\sigma\sigma(a_0 b_0)(r - c_0) + 11b\sigma(a_0 b_0)(r - c_0) - 3(a_0 b_0)a_0^2 - 10\sigma(a_0 b_0)a_0^2$$
$$- 2b(a_0 b_0)a_0^2 - 14\sigma b(a_0 b_0)c_0 + 11\sigma(a_0 b_0)^2 - (b c_0) - 2\sigma(b c_0) - b(b c_0)$$
$$- 3\sigma^2(b c_0) - 3\sigma b(b c_0) - b^2(b c_0) - 4\sigma^3(b c_0) - 6\sigma^2 b(b c_0) - 4\sigma b^2(b c_0)$$
$$- b^3(b c_0) - 9\sigma(b c_0)(r - c_0) - 11\sigma\sigma(b c_0)(r - c_0) - 7b\sigma(b c_0)(r - c_0)$$
$$+ 2(b c_0)a_0^2 + 9\sigma(b c_0)a_0^2 + 2b(b c_0)a_0^2 + 4(\sigma b c_0)^2 - 3a_0^2(r - c_0)$$
$$- 14\sigma a_0^2(r - c_0) - 2b a_0^2(r - c_0) - 25\sigma^2 a_0^2(r - c_0) - 7\sigma b a_0^2(r - c_0)$$
$$- b^2 a_0^2(r - c_0) - 11\sigma a_0^2(r - c_0)^2 + a_0^4(r - c_0) - 11(\sigma b_0^2) - 20\sigma(\sigma b_0^2)$$
$$- 4b(\sigma b_0^2) - 11\sigma^2(\sigma b_0^2) - 5\sigma b(\sigma b_0^2) - b^2(\sigma b_0^2) - 11\sigma(\sigma b_0^2)(r - c_0)\};$$

$$\hat{\hat{E}}_5^{n,n} = \{-22(\sigma a_0 b_0) - 30\sigma(\sigma a_0 b_0) - 13b(\sigma a_0 b_0) - 14\sigma^2(\sigma a_0 b_0) - 10\sigma b(\sigma a_0 b_0)$$
$$- 4b^2(\sigma a_0 b_0) - 22\sigma(\sigma a_0 b_0)(r - c_0) + 4(\sigma a_0 b_0)a_0^2 + 9(\sigma b c_0) + 7\sigma(\sigma b c_0)$$
$$+ 9b(\sigma b c_0) + 4\sigma^2(\sigma b c_0) + 6\sigma b(\sigma b c_0) + 4b^2(\sigma b c_0) + 6\sigma(\sigma b c_0)(r - c_0)$$
$$- 4(\sigma b c_0)a_0^2 + 13\sigma a_0^2(r - c_0) + 18\sigma\sigma a_0^2(r - c_0) + 4b\sigma a_0^2(r - c_0)$$
$$+ 21(\sigma^2 b_0^2) + 10\sigma(\sigma^2 b_0^2) + 4b(\sigma^2 b_0^2)\};$$

$$\hat{\hat{E}}_6^{n,n} = \{-52(\sigma a_0 b_0) - 100\sigma(\sigma a_0 b_0) - 38b(\sigma a_0 b_0) - 84\sigma^2(\sigma a_0 b_0)$$
$$- 50\sigma b(\sigma a_0 b_0) - 19b^2(\sigma a_0 b_0) - 30\sigma^3(\sigma a_0 b_0) - 25\sigma^2 b(\sigma a_0 b_0)$$
$$- 15\sigma b^2(\sigma a_0 b_0) - 5b^3(\sigma a_0 b_0) - 150\sigma(\sigma a_0 b_0)(r - c_0)$$
$$- 128\sigma^2(\sigma a_0 b_0)(r - c_0) - 33\sigma b(\sigma a_0 b_0)(r - c_0) + 24(\sigma a_0 b_0)a_0^2$$
$$+ 40\sigma(\sigma a_0 b_0)a_0^2 + 10b(\sigma a_0 b_0)a_0^2 - 45\sigma a_0 b_0 \sigma a_0 b_0 + 50\sigma a_0 b_0 \sigma b c_0$$
$$+ 14(\sigma b c_0) + 12\sigma(\sigma b c_0) + 19b(\sigma b c_0) + 9\sigma^2(\sigma b c_0) + 16\sigma b(\sigma b c_0)$$
$$+ 14b^2(\sigma b c_0) + 5\sigma^3(\sigma b c_0) + 10\sigma^2 b(\sigma b c_0) + 10\sigma b^2(\sigma b c_0) + 5b^3(\sigma b c_0)$$
$$+ 21\sigma(\sigma b c_0)(r - c_0) + 15\sigma^2(\sigma b c_0)(r - c_0) + 13\sigma b(\sigma b c_0)(r - c_0)$$

$$-19(\sigma bc_0)a_0^2-35\sigma(\sigma bc_0)a_0^2-10b(\sigma bc_0)a_0^2-10\sigma bc_0\sigma bc_0+96(\sigma^2 b_0^2)$$
$$+80\sigma(\sigma^2 b_0^2)+29b(\sigma^2 b_0^2)+25\sigma^2(\sigma^2 b_0^2)+15\sigma b(\sigma^2 b_0^2)+5b^2(\sigma^2 b_0^2)$$
$$+33\sigma(\sigma^2 b_0^2)(r-c_0)+38\sigma a_0^2(r-c_0)+88\sigma\sigma\sigma a_0^2(r-c_0)+19b\sigma a_0^2(r-c_0)$$
$$+75\sigma^2\sigma a_0^2(r-c_0)+25\sigma b\sigma a_0^2(r-c_0)+5b^2\sigma a_0^2(r-c_0)$$
$$+33\sigma\sigma a_0^2(r-c_0)(r-c_0)-5\sigma a_0^4(r-c_0)\}.$$

For $j\in\{1,2,3,4,5,6\}$ we calculate from (1.3) the exact values of b_j and we set $b_j(\approx)=b_j$ fro $j\in\{0,1,2,3,4,5,6\}$.

5. LOCAL APPROXIMATIONS

For parameters σ, r, b. and initial values a_0, b_0, c_0 we have the following power series:

$$a_0(\approx)+a_1(\approx)t+a_2(\approx)\frac{t^2}{2!}+\ldots+a_n(\approx)\frac{t^n}{n!}+\ldots=\sum_{n=0}^{\infty}a_n(\approx)\frac{t^n}{n!}$$
$$b_0(\approx)+b_1(\approx)t+b_2(\approx)\frac{t^2}{2!}+\ldots+b_n(\approx)\frac{t^n}{n!}+\ldots=\sum_{n=0}^{\infty}b_n(\approx)\frac{t^n}{n!} \qquad (5.1)$$
$$c_0(\approx)+c_1(\approx)t+c_2(\approx)\frac{t^2}{2!}+\ldots+c_n(\approx)\frac{t^n}{n!}+\ldots=\sum_{n=0}^{\infty}c_n(\approx)\frac{t^n}{n!}.$$

At this moment the following question is open.

Question: What conditions would imply the convergence of the power series (5.1).

Next we turn our attention to the following polynomials, (5.2).

$$P_m(a_0,b_0,c_0)(t)=a_0(\approx)+a_1(\approx)t+\ldots+a_m(\approx)\frac{t^m}{m!}=\sum_{n=0}^{m}a_n(\approx)\frac{t^n}{n!}$$
$$Q_m(a_0,b_0,c_0)(t)=b_0(\approx)+b_1(\approx)t+\ldots+b_m(\approx)\frac{t^m}{m!}=\sum_{n=0}^{m}b_n(\approx)\frac{t^n}{n!} \qquad (5.2)$$
$$R_m(a_0,b_0,c_0)(t)=c_0(\approx)+c_1(\approx)t+\ldots+c_m(\approx)\frac{t^m}{m!}=\sum_{n=0}^{m}c_n(\approx)\frac{t^n}{n!}.$$

Let T be a positive real number, and m be a positive integer. For parameters σ, r, b and initial a_0, b_0, c_0 we define functions $x_T(t), y_T(t), z_T(t)$ for $t \in [0,\infty)$, as follows.

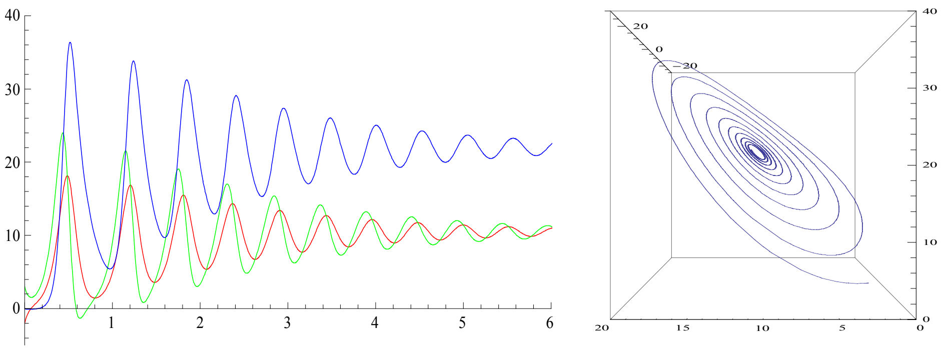
For $t \in [0,T]$, $x_T(t) = P_m(a_0,b_0,c_0)(t)$, $y_T(t) = Q_m(a_0,b_0,c_0)(t)$ and $z_T(t) = R_m(a_0,b_0,c_0)(t)$.

Next, continue by induction. Assume that $x_T(t), y_T(t), z_T(t)$ are defined for $t \in [0,k\cdot T]$. We define them for $t \in [0,(k+1)\cdot T]$ as follows. For $t \in [0,k\cdot T]$ they are already defined, and for $t \in [k\cdot T,(k+1)\cdot T]$ we define them by: $x_T(t) = P_m(u_0,v_0,w_0)(t-kT)$, $y_T(t) = Q_m(u_0,v_0,w_0)(t-kT)$ and $z_T(t) = R_m(u_0,v_0,w_0)(t-kT)$, where $u_0 = x_T(kT)$, $v_0 = y_T(kT)$ and $w_0 = x_T(kT)$.

At this moment we do not have the answer to the question of how good approximations, modulo T and m, are the functions $x_T(t), y_T(t), z_T(t)$ for the solutions of the Lorenz system (1.1).

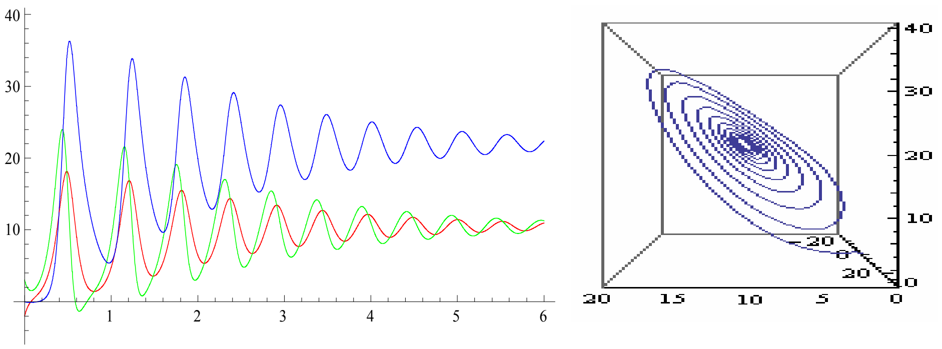
In examples, by computer calculations, for small values of T, we obtain that the functions $x_T(t), y_T(t), z_T(t)$ are good approximations for the solutions of the system (1.1). We used the program Mathematica and compared the solutions obtained by the program Mathematica with the functions $x_T(t), y_T(t), z_T(t)$. Two of these examples are the following.

Example 1: Parameters: $\sigma = 10, r = 23, b = 5$; initial values $a_0 = -2$, $b_0 = 3, c_0 = 0$; T=0,05; m=20; and the time interval [0,6].



a) Graphics of the solutions **b)** Space curve solution

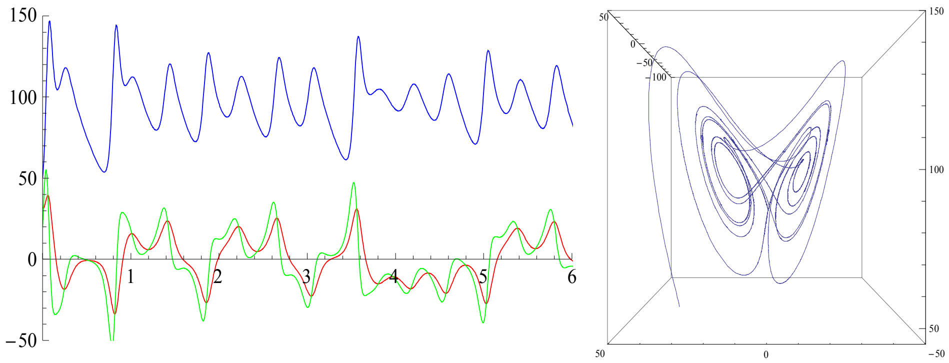
Fig. 1. Results obtained by the program Mathematica



a) Graphics of the functions $x_T(t), y_T(t), z_T(t)$ **b)** Space curve $(x_T(t), y_T(t), z_T(t))$

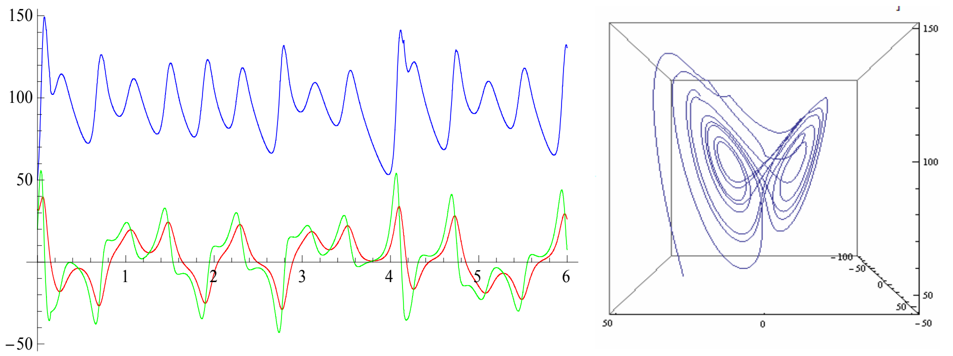
Fig. 2. Results obtained by computation

Example 2: Parameters: $\sigma = 12, r = 100, b = 2$; initial values $a_0 = 32$, $b_0 = 20, c_0 = 50$; T=0,05; m=20; and the time interval [0,6].



a) Graphics of the solutions **b)** Space curve solution

Fig. 3. Results obtained by the program Mathematica



a) Graphics of the functions $x_T(t), y_T(t), z_T(t)$ **b)** Space curve $(x_T(t), y_T(t), z_T(t))$

Fig. 4. Results obtained by computation

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СИСТЕМИ ОД ДИФЕРЕНЦНИ РАВЕНКИ КАКО АПРОКСИМАЦИЈА НА ЛОРЕНЦОВИОТ СИСТЕМ ОД ДИФЕРЕНЦИЈАЛНИ РАВЕНКИ

Во овој труд од Лоренцовиот систем диференцијални равенки се добиени системи диференцни равенки. Користејќи некои регуларности во овие ситеми од диференцни равенки, добиени се полиномни апроксимации на нивните решенија. Земајќи ги овие апроксимации како коефициенти, добиени се три степенски реда и со компјутерски пресметки е проверувано дека тие даваат локална апроксимација на решенијата на Лоренцовиот систем диференцијални равенки.

Клучни зборови: Лоренцов систем; диференцијални равенки; диференцни равенки; степенски редови; апроксимација

Address:

Biljana Zlatanovska
University "Goce Delčev", Faculty of Computer Science
Štip,Republic of Macedonia
zlatanovskabiljana_1973@yahoo.com

Dončo Dimovski
Ss. Cyril and Methodius University in Skopje
Faculty of Natural Sciences and Mathematics,
PO Box 162, MK-1001 Skopje, Republic of Macedonia
donco@pmf.ukim.mk