ПРИЛОЗИ, Одделение за природно-математички и биотехнолошки науки, том **41**, бр. 2, стр. 115–120 (2020)
CONTRIBUTIONS, Section of Natural, Mathematical and Biotechnical Sciences, MASA, Vol. **41**, No. 2, pp. 115–120 (2020)

Received: December 6, 2020
Accepted: December 19, 2020

ISSN 1857-9027
e-ISSN 1857-9949
UDC: 512.72
DOI: 10.20903/csnmbs.masa.2020.41.2.161

Original scientific paper

LACONIC VARIETIES AND THE MEMBERSHIP PROBLEM

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To the memory of Professor Gjorgji Čupona

We introduce laconic varieties and algebras, inspired by a closely related notion in monoids. After providing basic properties of laconic algebras, we define upper distortion functions for laconic subalgebras and apply it to the Membership Problem.

Key words: Membership Problem, distortion function, laconic variety

INTRODUCTION

Graded monoids were introduced in [8] by Margolis, Meakin, and the author as a tool for proving the decidability of certain instances of the Membership Problem in submonoids of groups, which in turn, by earlier results of Ivanov, Margolis, and Meakin [6], implied the decidability of the Word Problem in certain one-relator inverse monoids. Recently, Silva and Zakharov [10] used graded monoids in relation to algorithmic problems in virtually free groups. The notion we are introducing here, laconic algebra, is not exactly a generalization of the notion of a graded monoid to other varieties, but it is closely modeled on it. One advantage of the slightly changed approach is that the new notion is independent of the choice of generating sets (for graded monoids one had to be careful not to include the identity in the generating set), which makes some of the discussion smoother. On the other hand, there are no laconic monoids, so something is lost in this exchange too. The idea behind the approach is very simple – in some algebras, one can tell that some elements cannot be equal just by looking at the lengths of the terms that represent them.

After introducing laconic algebras and varieties, and providing some basic general properties in Section 2, we define upper distortion functions in Section 3, which are related to the corresponding notion in graded monoids, and show how upper distortion can be applied to solve some instances of the Membership Problem. We end with a simple example.

DEFINITION AND BASIC PROPERTIES

We start by defining laconic algebras and varieties, and establishing some of their basic properties.

Definition 2.1 (Laconic algebra/variety)**.** Let $\mathcal{V}$ be a variety. An algebra $\mathbf{A}$ in $\mathcal{V}$ is *laconic* if, for every free algebra $\mathbf{F}$ of finite rank in $\mathcal{V}$, every homomorphism $\phi : \mathbf{F} \to \mathbf{A}$, and every element a in A, the fiber $\phi^{-1}(a)$ is finite.

The variety $\mathcal{V}$ is *laconic* if it contains at least one nonempty laconic algebra.

Recall that, in a variety without constants, the free algebra of rank 0 is the empty algebra, which is, vacuously, laconic. This (and other reasons) is why the definition insists on the existence of a nonempty laconic algebra.

Example 2.1. The variety of semigroups is laconic. Consider the free semigroup of rank 1, namely $\mathbb{N}^+$. Let X be finite, $\phi : X^+ \to \mathbb{N}^+$ a homomorphism from the free semigroup X^+ to $\mathbb{N}^+$, and $a \in \mathbb{N}^+$. No words over X of length larger than a can be mapped to a under ϕ. Since there are only finitely many words over X of length at most a, the fiber $\phi^{-1}(a)$ contains only finitely many words. Thus, $\mathbb{N}^+$ is a laconic semigroup.

This example indicates why we use the term laconic – any element in a laconic semigroup is represented by only a few (finitely many) words.

The variety of monoids is not laconic. There are laconic varieties with constants. For instance, the variety of semigroups with a "central constant" c, defined by the identities $x(yz) = (xy)z$ and $cx = xc$, is laconic.

Example 2.2. Many of the varieties of grupoids studied by Čupona, his collaborators, Celakoski, Dimovski, Markovski, Janeva, Ilić, and their students, are laconic. For instance, the variety of grupoids defined by a single identity of the form $(xy)^n = x^n y^n$, studied in [2, 4], is laconic, and so are the varieties of monoassociative and biassociative grupoids [3, 5].

Example 2.3. In fact, any variety defined by balanced identities is laconic. In such a variety, the free algebra $\mathbf{F}_1$ of rank 1 is laconic, since any element of length at least k in any term algebra maps to an element of (term) length at least k in $\mathbf{F}_1$ (more on term algebras and lengths later).

Example 2.4. The identities defining laconic varieties do not need to be balanced. For instance, the variety of left zeros, defined by the identity $xy = x$, is laconic. In fact, all algebras in this variety are laconic, since all algebras in this variety are free, all maps between them are homomorphisms, and the f.g. free algebras are precisely the finite ones. Related examples of laconic varieties with non-balanced identities are the varieties of k-left-zero semigroups ($k \geq 0$), studied in [9]. For a laconic variety of semigroups, only these two options are available: either it is defined by balanced identities or all of its f.g. algebras are finite. There are laconic varieties of grupoids with non-balanced identities and infinite f.g. free algebras.

Proposition 2.1 (Closure properties)**.** *Let $\mathcal{V}$ be a laconic variety.*

(a) The subclass of laconic algebras in $\mathcal{V}$ is closed under subalgebras.

(b) The subclass of laconic algebras in $\mathcal{V}$ is closed under inverse images.

(c) The subclass of laconic algebras in $\mathcal{V}$ is closed under arbitrary products.

(d) Any product in $\mathcal{V}$ in which at least one factor is laconic is itself laconic.

(e) All free algebras in $\mathcal{V}$ are laconic.

(f) If $\mathbf{A}$ is laconic and there exists a homomorphism $\psi : \mathbf{B} \to \mathbf{A}$, then $\mathbf{B}$ is laconic.

Proof. (a) Let $\mathbf{A}$ be a laconic algebra and $\mathbf{B} \leq \mathbf{A}$. Any homomorphism $\phi : \mathbf{F} \to \mathbf{B}$ from a free algebra $\mathbf{F}$ of finite rank to $\mathbf{B}$ is a restriction (in codomain) of the homomorphism $\phi' : \mathbf{F} \to \mathbf{A}$, where, for all $f \in F$, we have $\phi'(f) = \phi(f)$. Every ϕ-fiber of an element in B is a ϕ'-fiber, and since $\mathbf{A}$ is laconic, any such fiber is finite. Thus $\mathbf{B}$ is laconic.

(b), (c), (d), and (e) are corollaries of (f).

(f) Let $\phi : \mathbf{F} \to \mathbf{B}$ be a homomorphism from the free algebra $\mathbf{F}$ of finite rank to $\mathbf{B}$. Since $\mathbf{A}$ is laconic, all fibers of the homomorphism $\psi\phi : \mathbf{F} \to \mathbf{A}$ are finite. For any element b in B, the ϕ-fiber of b is a subset of the $\psi\phi$-fiber of $\psi(b)$, which is finite. Thus, $\mathbf{B}$ is laconic. □

Corollary 2.2. *A variety $\mathcal{V}$ is laconic if and only it its free algebra of rank 1 is laconic.*

The property of being laconic is local.

Proposition 2.3 (Laconic is local)**.** *An algebra $\mathbf{A}$ in a laconic variety $\mathcal{V}$ is laconic if and only if every finitely generated subalgebra of $\mathbf{A}$ is laconic.*

Proof. For the forward direction, recall that the class of laconic algebras is closed under subalgebras.

For the backward direction, assume that all finitely generated subalgebras of $\mathbf{A}$ are laconic. Let $\phi : \mathbf{F} \to \mathbf{A}$ be a homomorphism from the free algebra $\mathbf{F}$ of finite rank to $\mathbf{A}$. Since $\mathbf{F}$ has finite rank, the subalgebra $\phi(\mathbf{F})$ of $\mathbf{A}$ is finitely generated, which implies that $\phi(\mathbf{F})$ is laconic. Every ϕ-fiber of an element in A is either empty or a fiber of an element in $\phi(\mathbf{F})$. In both cases, the fiber is finite. Thus, $\mathbf{A}$ is laconic. □

Corollary 2.4. *The class of laconic algebras in a laconic variety $\mathcal{V}$ is closed under directed unions.*

The subclass of laconic algebras in a laconic variety is not, in general, closed under homomorphic images. For instance, finite semigroups are not laconic, but they are images of free semigroups, which are laconic. However, the property is preserved under homomorphic images, provided the fibers of the homomorphism are finite.

Proposition 2.5 (Laconic images)**.** *An algebra $\mathbf{A}$ in a laconic variety $\mathcal{V}$ is laconic if and only if it is a homomorphic image, with finite fibers, of a laconic algebra.*

Proof. For the forward direction, observe that the identity map has finite fibers.

For the backward direction, assume that **B** is a laconic algebra and $\psi : \mathbf{B} \to \mathbf{A}$ is a surjective homomorphism with finite fibers. Let $\phi : \mathbf{F} \to \mathbf{A}$ be a homomorphism from the free algebra **F** of finite rank to **A**. By the projective property of the free algebra **F**, there exist a lift $\phi' : \mathbf{F} \to \mathbf{B}$, such that $\phi = \psi\phi'$. The fibers of ψ are finite by assumption, and the fibers of ϕ' are finite, since **B** is laconic. Thus, **A** is laconic. □

UPPER DISTORTION AND APPLICATION TO THE MEMBERSHIP PROBLEM

In this section we discuss algorithmic issues and, accordingly, limit our attention to finitely generated algebras in laconic varieties of finite type. Parts of the discussion are valid in wider settings, but we will not attempt to indicate such moments.

Let $\mathcal{V}$ be any variety of finite type and X a finite set. A general way to construct the f.g. free algebra $\mathbf{F}(X)$ in $\mathcal{V}$ is by using the set $T(X)$ of terms over X, and the corresponding term algebra $\mathbf{T}(X)$ (see [1]). The elements of $\mathbf{T}(X)$ are classes of terms that are identified by the identities of $\mathcal{V}$. For a term τ in $T(X)$, the element of $\mathbf{T}(X)$ represented by τ is denoted by $\overline{\tau}$. The length of a term τ in $T(X)$, denoted $|\tau|_X$, is the total number occurrences of k-ary operation symbols, for $k \geq 1$ (symbols for constants are not counted). To emphasize the dependence on X, we sometimes call this length the X-length and we say X-term for an element of $T(X)$ (especially when there are other term algebras and bases around). The set of all X-terms of X-length no greater than n is denoted by $T_n(X)$. The length of an element $\overline{\tau}$ in the term algebra $\mathbf{T}(X)$, denoted $|\overline{\tau}|_X$, is the length of the shortest term in the class of τ. The set of all elements in the term algebra $\mathbf{T}(X)$ of length no greater than n is denoted by $\mathbf{T}_n(X)$. Since X and the type are finite, both $T_n(X)$ and $\mathbf{T}_n(X)$ are finite and, for future reference, we note that $\overline{T_n(X)} = \mathbf{T}_n(X)$.

Let **A** be a f.g. algebra in the variety $\mathcal{V}$. One of the ways to give a representation of the algebra **A** is through a surjective homomorphism $\psi : \mathbf{T}(X) \to \mathbf{A}$ from a term algebra $\mathbf{T}(X)$ over a finite basis X, along with a description of the corresponding congruence θ on $\mathbf{T}(X)$ such that $\mathbf{T}(X)/\theta \cong \psi(\mathbf{T}(X)) = \mathbf{A}$. Concretely, if we are given a finite set R of pairs in $\mathbf{T}(X)$ that generates the congruence θ, we say that the algebra **A** is finitely presented by the pair (X, R). The Word Problem for the finite presentation of **A** given by (X, R) asks for an algorithm deciding, for any two terms τ_1 and τ_2 in $T(X)$, if $\overline{\tau}_1\theta\overline{\tau}_2$, that is, if $\psi(\overline{\tau}_1) = \psi(\overline{\tau}_2)$. We take a more general view of the Word Problem as follows. The elements of the algebra **A** may be represented in any particular way (sets, functions, diagrams, graphs, matrices, or any other convenient construction). Note that defining ϕ amounts to naming a finite generating system for **A** (we say system rather than set, since we may choose, on purpose or unknowingly, the same element from A several times in the system). The Word Problem then asks for an algorithm deciding, given any two terms τ_1 and τ_2 in $T(X)$, if $\psi(\overline{\tau}_1) = \psi(\overline{\tau}_2)$. When such an algorithm exists, we say that the Word Problem for **A** is decidable.

Let a f.g. subalgebra **B** of the f.g. algebra **A** be given by a finite set T of terms in $\mathbf{T}(X)$ such that $\psi(T)$ generates **B**. The Membership Problem for **B** in **A** asks for an algorithm deciding, given any term τ in $T(X)$, if $\psi(\overline{\tau}) \in B$. When such an algorithm exists, we say that the Membership Problem for **B** in **A** is decidable. It is known that the decidability of the Word Problem and the Membership Problem do not depend on the choice of the homomorphism ψ (they are properties of the algebras, not of the representations).

Standing assumptions. We make several standing assumptions.

We consider two varieties $\mathcal{W}$ and $\mathcal{V}$ of finite types $\Omega_{\mathcal{W}}$ and $\Omega_{\mathcal{V}}$, respectively, such that $\Omega_{\mathcal{W}} \supseteq \Omega_{\mathcal{V}}$, the set of identities of $\mathcal{W}$ includes those of $\mathcal{V}$, and $\mathcal{V}$ is laconic (a simple example to have in mind: $\mathcal{W}$ is the variety of groups and $\mathcal{V}$ is the variety of semigroups). Let **A** be a f.g. $\mathcal{W}$-algebra, X a finite set, $\psi : \mathbf{T}(X) \to \mathbf{A}$ a representation of **A**, and $T = \{\tau_1, \dots, \tau_m\}$ a finite set of X-terms. Since **A** can also be considered as a $\mathcal{V}$-algebra, we can consider the $\mathcal{V}$-subalgebra of **A** given by $\mathbf{B} = \langle\psi(\tau_1), ..., \psi(\tau_m)\rangle_{\mathcal{V}}$. Let $Y = \{y_1, \dots, y_m\}$, with the obvious bijection to T, and define a representation $\phi : \mathbf{T}(Y) \to \mathbf{B}$ by $\phi(y_i) = \psi(\tau_i)$, for $i = 1, \dots, m$.

We are interested in the Membership Problem for **B** in **A**, that is, given arbitrary $\tau \in T(X)$, we want to know if $\psi(\tau) \in B$. In general, the terms in $T(X)$ are of type $\Omega_{\mathcal{W}}$ and those in $T(Y)$ are of type $\Omega_{\mathcal{V}}$. Thus, the terms $\tau.\tau_1, \dots, \tau_m$ may use operation symbols that are not in $\Omega_{\mathcal{V}}$ and we have a slightly extended view of the Membership Problem, which in its standard setting has $\mathcal{W} = \mathcal{V}$.

Definition 3.1 (Upper distortion)**.** Standing assumptions apply. If $\mathbf{B}$ is laconic, the *actual upper distortion function* for $\mathbf{B}$ in $\mathbf{A}$, with respect to ψ and ϕ, is the function $\hat{f} : \mathbb{N} \to \mathbb{N}$ defined by

$$\hat{f}(n) = \max\{\ |\bar{t}|_Y \ :\ t \in T(Y),\ \bar{t} \in \phi^{-1}\psi(\mathbf{T}_n(X))\ \}.$$

An *upper distortion function* for $\mathbf{B}$ in $\mathbf{A}$ is any function $f : \mathbb{N} \to \mathbb{N}$ that bounds the actual distortion function from above.

Let us quickly verify that the definition of the actual upper distortion function $\hat{f}$ makes sense. The set $\mathbf{T}_n(X)$ is finite, which makes $\psi(\mathbf{T}_n(X))$ finite as well. Since $\mathbf{B}$ is laconic the set $\phi^{-1}\psi(\mathbf{T}_n(X))$ is finite, which means that the maximum exists.

For better understanding, let us also parse the meaning of any upper distortion function f. The set $\psi(\mathbf{T}_n(X)) = \psi(\overline{T_n(X)})$ is the finite set of elements in A that can be represented by an X-term of X-length no greater than n. The set $\phi^{-1}\psi(\mathbf{T}_n(X))$ is then the finite set of all elements in the term algebra $\mathbf{T}(Y)$ that represent the elements in $B \cap \psi(\overline{T_n(X)})$. Since none of the elements in $\phi^{-1}\psi(\mathbf{T}_n(X))$ has Y-length greater than $\hat{f}(n) \leq f(n)$, we have

$$B \cap \psi(\overline{T_n(X)}) = \phi\phi^{-1}\psi(\mathbf{T}_n(X)) \subseteq \phi(\mathbf{T}_{f(n)}(Y)) = \phi(\overline{T_{f(n)}(Y)}).$$

In other words, every element of $\mathbf{B}$, representable by an X-term of length at most n, must be representable by a Y-term of length at most $f(n)$. We could say the upper distortion gives an upper bound on the "distortion in length" from a representation of the elements in $\mathbf{B}$ by X-terms (external generators, operation symbols in $\Omega_{\mathcal{W}}$) to a representation by Y-terms (internal generators for $\mathbf{B}$, operation symbols from $\Omega_{\mathcal{V}}$). With this understanding the next results is practically a tautology.

Proposition 3.1 (Membership Problem)**.** *Standing assumptions apply. Assume further that the Word Problem for* $\mathbf{A}$ *(as a* $\mathcal{W}$*-algebra) is decidable,* $\mathbf{B}$ *is laconic, and there is a computable (recursive) upper distortion function* f *for* $\mathbf{B}$ *in* $\mathbf{A}$ *with respect to* ψ *and* ϕ*. Then, the Membership Problem for* $\mathbf{B}$ *in* $\mathbf{A}$ *is decidable.*

Proof. We present an algorithm solving the Membership Problem.

Because Y and the type $\Omega_{\mathcal{V}}$ are finite, we may list all Y-terms by length (first all with length 0, then those with length 1, and so on). For every Y-term $t(y_1, \dots, y_m)$ in this list, we have

$$\phi(t(\overline{y}_1, \dots, \overline{y}_m)) = t(\phi(\overline{y}_1), \dots, \phi(\overline{y}_m)) = t(\psi(\overline{\tau}_1), \dots, \psi(\overline{\tau}_m)) = \psi(t(\overline{\tau}_1, \dots, \overline{\tau}_m)),$$

that is, the Y-term $t(y_1, \dots, y_m)$ represents the same element in B as the X-term $t(\tau_1, \dots, \tau_m)$ does. For every term $t(y_1, \dots, y_m)$ in the list of Y-terms ordered by length, consider the corresponding X-term $t(\tau_1, \dots, \tau_m)$. We can, by the decidability of the Word Problem for $\mathbf{A}$, decide if $t(\tau_1, \dots, \tau_m)$ and τ represent the same element of A. If, at any point, the answer is yes, we may stop and declare that $\psi(\overline{\tau})$ is in B. Assume that the X-length of τ is n. Once we check all terms in $T_{f(n)}(Y)$ and if we still do not have a positive answer, we may stop and declare that $\psi(\overline{\tau})$ is not in B. Indeed, if $\psi(\overline{\tau}) \in B$, then

$$\psi(\overline{\tau}) \in B \cap \psi(\overline{T_n(X)}) \subseteq \phi(\overline{T_{f(n)}(Y)}),$$

which means that, once we verify that $\psi(\overline{\tau}) \notin \phi(\overline{T_{f(n)}(Y)})$, we know that $\psi(\overline{\tau}) \notin B$. □

The previous proposition seems difficult to use, since it is not always clear how one can find an upper distortion function. The following proposition says that if one understands a laconic homomorphic image, which is presumably simpler and easier for analysis, one can just lift any upper distortion function found for the image and use it.

Proposition 3.2 (Lifting)**.** *Standing assumptions apply. Let* $\alpha : \mathbf{A} \to \mathbf{A}'$ *be a surjective* $\mathcal{W}$ *homomorphism,* $\alpha_B : \mathbf{B} \to \mathbf{B}'$ *its restriction to a surjective* $\mathcal{V}$*-homomorphism, where* $B' = \alpha(B) = \alpha_B(B)$*. The term algebra* $\mathbf{T}(X)$ *represents the elements of* $\mathbf{A}'$ *through* $\alpha\psi$ *and the term algebra* $\mathbf{T}(Y)$ *represents the elements of* $\mathbf{B}'$ *through* $\alpha_B\phi$*. If* $\mathbf{B}'$ *is laconic, so is* $\mathbf{B}$*, and any upper distortion function* f' *for* $\mathbf{B}'$ *in* $\mathbf{A}'$*, with respect to* $\alpha\psi$ *and* $\alpha_B\phi$*, is an upper distortion function for* $\mathbf{B}$ *in* $\mathbf{A}$*, with respect to* ψ *and* ϕ*.*

Proof. The algebra $\mathbf{B}$ is laconic as an inverse image of the laconic algebra $\mathbf{B}'$. Let $\bar{t}$ be an element of the term algebra $\mathbf{T}(Y)$. We have

$$\begin{aligned} & \bar{t} \in \phi^{-1}\psi(\mathbf{T}_n(X)) \\ \Longrightarrow\ & \phi(\bar{t}) \in (B \cap \psi(\mathbf{T}_n(X))) \\ \Longrightarrow\ & \alpha_B\phi(\bar{t}) \in \alpha\psi(\mathbf{T}_n(X)) \\ \Longrightarrow\ & \bar{t} \in (\alpha_B\phi)^{-1}(\alpha\psi)(\mathbf{T}_n(X)), \end{aligned}$$

which shows that $\phi^{-1}\psi(\mathbf{T}_n(X)) \subseteq (\alpha_B\phi)^{-1}(\alpha\psi)(\mathbf{T}_n(X))$ and, therefore $\hat{f}(n)$, the maximum length of an element in $\phi^{-1}\psi(\mathbf{T}_n(X))$, is smaller than or equal to $\hat{f}'(n)$, the maximum length of an element in $(\alpha_B\phi)^{-1}(\alpha\psi)(\mathbf{T}_n(X))$. Thus, for any upper distortion function f' for $\mathbf{B}'$ in $\mathbf{A}'$, we have $\hat{f} \leq \hat{f}' \leq f'$. □

Our final result provides a way to adapt a given upper distortion function from one representation to another. First a simple observation is in order. Let $\psi : \mathbf{T}(X) \to \mathbf{A}$ and $\psi' : \mathbf{T}(X') \to \mathbf{A}$ be two representations of $\mathbf{A}$. Let M be the smallest number such that, for each letter $x' \in X'$, there exists an X-term $\tau_{x'}$ of length at most M such that $\psi(\overline{\tau_{x'}}) = \psi'(\overline{x'})$ (such an M must exist, since X' is finite). Let K be the largest arity of a symbol in $\Omega_{\mathcal{W}}$. Then, for any X'-term τ' of length at most n, there exists an X-term of length at most $(M(K-1)+1)n + M$ that represents the same element in $\mathbf{A}$ as $\psi'(\tau')$. In other words, there exists a linear function $g_{X',X}$ such that, for all n, we have $\psi'(\mathbf{T}_n(X')) \subseteq \psi(\mathbf{T}_{g_{X',X}(n)}(X))$. Analogous linear function exists for any rewriting from one representation to another (from one finite generating system to another).

Proposition 3.3 (Change of representation)**.** *Standing assumptions apply. Let $\psi' : \mathbf{T}(X') \to \mathbf{A}$ and $\phi' : \mathbf{T}(Y') \to \mathbf{B}$ be additional representations of $\mathbf{A}$ and $\mathbf{B}$, respectively, and let $\mathbf{B}$ be laconic. If f is an upper distortion function for $\mathbf{B}$ in $\mathbf{A}$ with respect to ψ and ϕ, then f', defined by $f'(n) = g_{Y,Y'}(f(g_{X',X}(n)))$, is an upper distortion function for $\mathbf{B}$ in $\mathbf{A}$ with respect to ψ' and ϕ'.*

Proof. For a term t' in $T(Y')$, if $\overline{t'} \in (\phi')^{-1}\psi'(\mathbf{T}_n(X'))$, then $\phi'(\overline{t'}) \in B \cap \psi'(\mathbf{T}_n(X'))$, which implies that

$$\phi'(\overline{t'}) \in B \cap \psi(\mathbf{T}_{g_{X',X}(n)}(X)) \subseteq \phi(\mathbf{T}_{f(g_{X',X}(n))}(Y)) \subseteq \phi'(\mathbf{T}_{g_{Y,Y'}f(g_{X',X}(n))}(Y')),$$

and this implies that $|t'|_Y \leq g_{Y,Y'}(f(g_{X',X}(n)))$. □

Example 3.1. Let $\mathcal{W}$ be the variety of groups, $\mathcal{V}$ the variety of semigroups, $X = \{x, y, z\}$, $Y = \{y_1, y_2, y_3\}$, $\mathbf{A}$ the group with presentation $\langle x, y, z \mid xy = zy^{-1}z^3x\rangle$, $\mathbf{B}$ the subsemigroup of $\mathbf{A}$ generated by $\{x, xy, y^3z^{-1}\}$, $\psi : \mathbf{T}(X) \to \mathbf{A}$ the obvious group representation of $\mathbf{A}$, and $\phi : \mathbf{T}(Y) \to \mathbf{B}$ the semigroup representations of $\mathbf{B}$ given by $\phi(y_1) = x$, $\phi(y_2) = xy$, $\phi(y_3) = y^3z^{-1}$. We want to solve the Membership Problem for $\mathbf{B}$ in $\mathbf{A}$.

Let $M_x = \left(\begin{smallmatrix}3 & 0\\ 0 & 1\end{smallmatrix}\right)$, $M_y = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$, $\mathbf{A}' = \langle M_x, M_y\rangle$, the subgroup of $SL_2(\mathbb{Z})$ generated by M_x and M_y, and α the surjective group homomorphism defined by $\alpha(x) = M_x$ and $\alpha(y) = \alpha(z) = M_y$. To verify that α defines a homomorphism we need to check that $M_xM_y = M_y^3M_x$, which does hold. Let $\mathbf{B}' = \alpha(\mathbf{B}) = \langle M_x, M_xM_y, M_y^2\rangle_{\mathcal{V}}$, that is, $\mathbf{B}'$ is the semigroup generated by the matrices $M_x = \left(\begin{smallmatrix}3 & 0\\ 0 & 1\end{smallmatrix}\right)$, $M_xM_y = \left(\begin{smallmatrix}3 & 3\\ 0 & 1\end{smallmatrix}\right)$ and $M_y^2 = \left(\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}\right)$. Let $X' = \{x, y\}$, $\psi' : \mathbf{T}(X') \to \mathbf{A}'$ be the group representation of $\mathbf{A}'$ given by $\psi'(x') = M_x$, $\psi'(y') = M_y$, $Y' = \{y_1', y_2', y_3'\}$, and $\phi' : \mathbf{T}(Y') \to \mathbf{B}'$ the semigroup representation of $\mathbf{B}'$ given by $\phi'(y_1') = M_x$, $\phi'(y_2') = M_xM_y$, $\phi'(y_3') = M_y^2$.

An easy induction on the length shows that if τ' is an X'-term of length at most n, and $\psi'(\overline{\tau}') = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, then $|a| + |b| \leq 3^{n+1}$. On the other hand, by induction on length, if t' is a Y' term of length at least n' and $\phi'(\overline{t}') = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, then $|a| + |b| > n'$. Therefore, $(\phi')^{-1}\psi'(\mathbf{T}_n(X')) \subseteq \mathbf{T}_{3^{n+1}}(Y')$. This shows that the fibers of ϕ' are finite. Since the fibers of ϕ' are finite and the free semigroup $\mathbf{T}(Y')$ is laconic, the semigroup $\mathbf{B}'$ is laconic by Proposition 2.5. Moreover, the function $f(n) = 3^{n+1}$ is an upper distortion function for $\mathbf{B}'$ in $\mathbf{A}'$ with respect to ψ' and ϕ'.

By Proposition 3.3 and the decidability of the Word Problem in one-relator groups [7], we can explicitly determine a computable upper distortion function for $\mathbf{B}'$ in $\mathbf{A}'$ with respect to $\alpha\psi$ and $\alpha_B\phi$, which we can lift, by Proposition 3.2, to an upper distortion function for $\mathbf{B}$ in $\mathbf{A}$ with respect to ψ and ϕ. Thus, by Proposition 3.1, the Membership Problem for $\mathbf{B}$ in $\mathbf{A}$ is decidable.

ACKNOWLEDGMENT

I would like to use this opportunity to acknowledge the crucial role Professor Čupona played in my mathematical growth. For four years, I was his student and teaching assistant and, as luck would have it, I got the priceless privilege of having constant access to his office, bookshelves, and most importantly, his thoughts, and I soaked in it all endlessly. Many things could be mentioned,

but let me limit myself to the following vignette. Almost thirty years ago, I gave my first talk at an international conference, in Potsdam, Germany. Right after the talk, an audience member, Professor Kaarli from University of Tartu, approached me and asked "Are you a student of Čupona?" I answered that I was and he simply said "You can always tell the lion by the trace he leaves." Never again in my career have I received a compliment that made me as happy. On behalf of all that still feel and cherish that trace, thank you Professor Čupona.

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ЛАКОНСКИ МНОГУОБРАЗИЈА И ПРОБЛЕМОТ НА ПРИПАДНОСТ

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Во спомен на професор Ѓорѓи Чупона

Ги воведуваме поимите на лаконски алгебри и многуобразија, инспирирани од близок поим кај моноидите. Откако ќе ги дадеме основните особини на лаконските алгебри, дефинираме функции на горна дисторзија на лаконските подалгебри и ги применуваме кон проблемот на припадност.

Клучни зборови: проблем на припадност, горна дисторзија, лаконско многуобразие

Би сакал да ја искористам оваа прилика да ја истакнам клучната улога што ја имаше професорот Чупона во мојот математички развој. Во тек на четири години, бев негов студент и асистент и, по среќна околност, се здобив со драгоцената привилегија да имам постојан пристап до неговата канцеларија, до полиците со книги и, најважно, до неговите мисли, и сето тоа безмерно го впивав. Многу нешта би можеле да се спомнат, но ќе се ограничам само на следнава вињета. Безмалку пред триесет години, го одржав своето прво предавање на меѓународна конференција, во Потсдам, Германија. Веднаш по предавањето, учесник на конференцијата, професор Каарли од Универзитетот во Тарту, пријде и ме праша „Дали сте студент на Чупона?“ Одговорив дека сум, на што тој само рече „Секогаш ќе го препознаете лавот по трагата што ја остава.“ Никогаш повторно во кариерата не добив комплимент што толку ме израдувал. Во името на сите што сè уште ја чувствуваат и ценат таа трага, благодарам професоре Чупона.