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Review

RECOGNIZABLE AND REGULAR SUBSETS OF MONOIDS

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To the memory of Professor Gjorgji Čupona, with gratitude

This note is a short review of regular and recognizable subsets of monoids. We introduce a new question about characterizing classes of monoids and show that idempotent monoids can be characterized by the properties of the languages they recognize.

Key words: monoids, automata, regular languages, recognizable languages

INTRODUCTION

It is well known that there is an intimate relationship between regular languages (as subsets of free monoids) and their syntactic monoids (as transition monoids of the corresponding minimal deterministic automata). Many classes of languages can be characterized by the ideal structure of their corresponding syntactic monoids [17]. In particular, classes of regular languages that are studied in symbolic dynamics and cellular automata can be characterized through their transition monoids of the minimal deterministic presentation [9, 11]. With this note, we ask the converse question: can properties of the languages recognized by classes of finite monoids describe, or characterize, the class of monoids? For the simple case of idempotent monoids we show that such characterization is possible.

The notions of automata and languages can be extended to arbitrary monoids. One can consider M-regular subsets of M where M is an arbitrary monoid, not necessarily the free monoid. Similarly, the recognizable languages can be extended to M-recognizable subsets. In this case, the M-regular subsets may strictly contain the M-recognizable subsets and we show why this inclusion is strict. We end by recalling the long standing open problem for characterizing the monoids for which M-recognizable and M-regular sets coincide.

PRELIMINARIES

2.1. **Automata.** A standard background in automata theory can be found in [8, 20]. A monoid with identity 1 is denoted with M. A subset of a monoid M is called an M-*language*. The set of all words over a finite alphabet A is denoted by A^*. With the operation concatenation A^* is the free monoid generated by A. A *language* is an A^*-language.

Definition 2.1 Let M be a monoid. An *M-finite state automaton* (or just *M-automaton*) is a tuple $\mathcal{M} = (M, Q, I, T, \mathcal{E})$ where Q is a finite set of *states*, $I \subseteq Q$ the set of *initial* states, $T \subseteq Q$ the set of *terminal* states and $\mathcal{E} \subseteq Q \times M \times Q$ the set of *transitions.*

An M-automaton $\mathcal{M}$ is associated with a finite labeled directed multigraph having vertices Q, directed edges $\mathcal{E}$, and three functions, $s, t : \mathcal{E} \to Q$ (*source* and *target* of

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the edges) and the labeling $\lambda : \mathcal{E} \to M$ defined by $s(q, a, q') = q$, $t(q, a, q') = q'$ and $(\lambda(q, a, q') = a$. A *transition sequence* or a *path* in $\mathcal{M}$ is a sequence of edges

$$\begin{aligned} p = \ & e_1 e_2 \cdots e_k \\ = \ & (q_0, a_1, q_1)(q_1, a_2, q_2) \cdots (q_{k-1}, a_k, q_k) \end{aligned}$$

satisfying $s(e_{i+1}) = t(e_i)$ for $i = 1, \ldots, k-1$. In fact $p \in \mathcal{E}^*$. The *label* of p is $\lambda(p) = \lambda(e_1) \cdots \lambda(e_k) = a_1 \cdots a_k \in M$. The *source* of p is $s(p) = s(e_1) = q_0$ and *target* of p is $t(p) = t(e_k) = q_k$.

A element $w \in M$ is *accepted* by $\mathcal{M}$ if there is a path p such that $s(p) \in I$, $t(p) \in T$ and $\lambda(p) = w$. In the case of $M = A^*$, w is called a word. The M-language *recognized by* $\mathcal{M}$ is $L(\mathcal{M}) = \{w \in M \mid w \text{ is accepted by } \mathcal{M}\}$. In particular, $1 \in L(\mathcal{M})$ if and only if $I \cap T \neq \emptyset$.

Definition 2.2 An M-language $L \subseteq M$ is *M-regular* if there exists an M-automaton $\mathcal{M}$ such that $L = L(\mathcal{M})$.

The class of M-regular languages is denoted $\mathcal{R}eg(M)$. Here we concentrate on *deterministic* M-automata, that is, for every $q \in Q$ and every $a \in M$ the set $\{q' \mid (q, a, q') \in \mathcal{E}\}$ is either a singleton or empty. It is well known that the class $\mathcal{R}eg(M)$ remains unchanged when we restrict our attention to deterministic automata. In the deterministic case, if $X \subseteq M$ is the set of labels of the transitions, then X^* (as a submonoid of M) acts on Q by $q \cdot a = q'$ or just $qa = q'$ for $(q, a, q') \in \mathcal{E}$ and $a \in X$. If there is no transition starting at q with label a then $qa = \emptyset$. One can always add a 'junk' state in Q and set $qa = junk$ whenever $qa = \emptyset$, hence, for each $a \in X$, its action on Q is considered as a function rather than a partial function. For $w \in X^*$, $qw = q'$ if there is a path in $\mathcal{M}$ from q to q' with label w. We usually take that $X^* = M$, i.e., X generates M. The *transition monoid* $\mathcal{T}(M)$ *of* $\mathcal{M}$ is the set X^* as functions acting on the states of $\mathcal{M}$.

2.2. **Monoids.** Let $L \subseteq M$ and $x \in M$. The *context of x in M with respect to L* is

$$C_L(x) = \{(u, v) \mid u, v, \in M, uxv \in L\}$$

We set $x \sim_L y$ if and only if $C_L(x) = C_L(y)$. The *syntactic semigroup* of L is the quotient $M/\sim_L$ denoted with $S(L)$ with the operation $[x][y] = [xy]$. A subset L of a monoid M is said to be *recognizable* if there is a morphism φ from M to a finite monoid N such that $L = \varphi^{-1}(P)$ for some subset $P \subseteq N$. A monoid N *recognizes* L if there is a morphism $\varphi : M \to N$ and a subset P of N such that $L = \varphi^{-1}(P)$. So L is recognizable if it is recognized by a finite monoid. The class of recognizable subsets of M is denoted by $\mathcal{R}ec(M)$. The following hold [17, 19].

Proposition 2.1

- *The syntactic monoid $S(L)$ recognizes L.*
- *If $L \in \mathcal{R}eg(A^*)$ then it is recognized by the transition monoid $\mathcal{T}(\mathcal{M})$ of the minimal deterministic $\mathcal{M}$. Moreover, $L \in \mathcal{R}eg(A^*)$ if and only if $L \in \mathcal{R}ec(A^*)$.*
- *A monoid M recognizes $L \subseteq A^*$ if and only if the syntactic monoid $S(L)$ of L divides M (it is a quotient of a submonoid of M).*

The ideal structure of a monoid can be described with the following equivalence relations which are based on the principal ideals.

Definition 2.3 Let M be a monoid. Green's relations R, L, J, H, D on M are defined as $(a, b \in M)$:

- $a\,R\,b$ if $aM = bM$
- $a\,L\,b$ if $Ma = Mb$
- $a\,J\,b$ if $MaM = MbM$
- $a\,H\,b$ if $a\,R\,b$ and $a\,L\,b$
- $a\,D\,b$ if there is $c \in M$ such that $a\,R\,c$ and $c\,L\,b$

In a finite monoid $D = J$. In this case the subgroups of M are the H classes containing idempotents.

MONOIDS AND LANGUAGES

3.1. **Monoid characterizations of classes of languages.** Algebraic characterization of languages is often used in automata theory, and some classes of languages show up in other fields, such as symbolic dynamics [14]. The concept of local languages remains fundamental in automata theory as every regular language is a morphic image of a local language (more precisely, strictly locally testable), and this characterization has been used to define regular 2-dimensional languages (sets of rectangular arrays of symbols [7, 12]). A language $L \subseteq A^*$ is *local* if it is a complement of finitely generated submonoid of A^*. In other words, L is local if there is a finite set of words F such that $L = A^* \setminus A^*FA^*$. The set F is called the set of *forbidden words*. A word $w \in A^*$ is called a *constant* for the language $L \subseteq A^*$ when for all $v_1, v_2, v_3, v_4 \in A^*$ the following implication holds,

$$v_1 w v_2 \in L \text{ and } v_3 w v_4 \in L \Rightarrow v_1 w v_4 \in L.$$

A well known characterization of local languages states: k is the maximal length of the set of forbidden words if and only if all words of length $\geq k$ are constants for L [14, 11]. These words also act as constant functions (hence the name) in the action of A^* on the minimal deterministic automaton $\mathcal{M}$ recognizing the language.

Definition 3.1 A language $\emptyset \neq L \subseteq A^*$ is:

- *factorial* if for all $x, y, z \in A^*$
  $xyz \in L \Rightarrow y \in L$
- *extendable* if for all $x \in L$
  there are $y, z \in A^+$ such that $yxz \in L$
- *transitive* for all $x, y \in L$
  there is $z \in A^*$ such that $xzy \in L$

Local, factorial and extendable languages correspond to factors of subshifts of finite type, while factorial, extendable and regular (FER) languages consist of factors of sofic subshifts. Transitive languages can be associated with transitive symbolic dynamical systems [14]. These languages, in particular factorial, transitive and regular (FTR) languages can also be studied as factors of images and traces of cellular automata [2]. Their syntactic monoids can be characterized as follows. We set $\eta : A^* \to S(L)$ as the natural onto morphism defined with $x \mapsto [x]$.

Proposition 3.1 [9] *L is an FTR language if and only if $S(L)$ has the following properties:*

(i) $S(L)$ is finite

(ii) $S(L)$ has a 0 such that $\eta^{-1}(S(L) - \{0\}) = L$

(iii) $S(L)$ has a 0-minimal right ideal R (an R-class) such that for every non zero $x \in S(L)$, $Rx \neq 0$.

In this case, one can define an A^*-automaton such that the states are the R-classes of the 0-minimal right ideal of $S(L)$ with transitions defined as $[x]a = [xa]$. This automaton, in fact, becomes the minimal transitive representation of the language [9]. Let $\mathcal{I}_c = \{[x] \,|\, x \text{ is a constant for } L\}$. Observe that $\mathcal{I}_c$ is an ideal for $S(L)$. Moreover, for a local, or an FTR-language L, the word $c \in L$ is constant if and only if $D_{[c]}$ has H-trivial subclasses, that is, the corresponding D-classes are group-free. This holds even for a larger class of languages that are factorial and extendable (not necessarily regular), such as the Dyck languages. Although their syntactic monoids are infinite, the classes of relations D and J coincide [10].

Proposition 3.2 [11, 15] *Let L be a language and $\mathcal{I}_c = \{[x] \,|\, x$ is a constant for $L\}$. The language L is local if and only if $[1] = \{1\}$ and the set of idempotents $E = \{e \,|\, e^2 = e, e \neq 1, e \neq 0\}$ is a non-empty subset of the ideal $\mathcal{I}_c$.*

Note that every finite group can be a syntactic monoid of some language by an appropriate definition of an action of the group to a directed graph.

3.2. **Language characterization of classes of monoids.** As computer science, and in particular, algebraic automata theory concentrates on understanding classes of languages, there has been virtually no studies of the converse question. *Characterize classes of monoids according to the classes of languages that they recognize.* In particular, consider the class $\mathcal{C}$ of finite monoids that belong to an identity defined variety of monoids, such as the variety defined by $x^n y^n = (xy)^n$ (varieties of such groupoids and other algebraic structures have been studied by Čupona and his collaborators, e.g. [3, 4]). Can the properties of the classes of languages that are recognized by $\mathcal{C}$ determine the monoids in $\mathcal{C}$? One simple case with a positive answer can be observed with the following. Recall that idempotent monoids are monoids whose every element is an idempotent.

Proposition 3.3 *A finite monoid M is an idempotent monoid if and only if every language L recognized by M satisfies the equivalence*

$$w \in L \Longleftrightarrow w^+ \subseteq L$$

Proof. If M is an idempotent monoid then M satisfies the equation $x^2 = x$. Let $\eta : A^* \to M$ be a morphism and L recognized through $P \subseteq M$ such that $L = \eta^{-1}(P)$. Then for $w \in L$, and $\eta(w) = p \in P$ we have that $pp = \eta(ww) = \eta(w^n) = p$, hence, for all n, $w^n \in \eta^{-1}(p)$, which implies that $w^+ \subseteq L$. Of course, if $w^+ \subseteq L$ then $w \in L$ by definition.

Converse, suppose that every L recognized by M satisfies the equivalence of the proposition. Consider a surjective morphism $\eta : A^* \to M$. Let $p \in M$ and take P to be the singleton $P = \{p\}$ and let $L = \eta^{-1}(p)$. Suppose w is of minimal length such that $\eta(w) = p$. The equivalence of the proposition says that $w^+ \subseteq L$ and hence $\eta(w^n) \in P$, i.e. $\eta(w^n) = p$. In particular, for $n = 2$ we have $\eta(ww) = pp = p$, implying that p must be an idempotent. As this is true for every $p \in M$, M is an idempotent monoid. □

For a finite idempotent monoid and $P \subseteq M$, let $\eta^{-1}(p)_{\min}$ be the minimal length $w \in A^*$ such that $\eta(w) = p$. Then whenever we have $p_1, p_2, p_1p_2 \in P$ the language recognized as $\eta\ ^{1}(P) \subseteq A^*$ contains the subsemigroup $(\eta^{-1}(p_1)_{\min} \cup \eta^{-1}(p_2)_{\min})^+$ which is extendable and transitive.

As a specific case, consider the idempotent monoid $\mathcal{J}_3 = \{1, h_1, h_2, h_1h_2, h_2h_1\}$ satisfying $h_ih_jh_i = h_i$ for $i, j = 1, 2$. It consists of a single D-class containing the four non-identity elements, two R-classes ($\{h_1, h_1h_2\}$, $\{h_2h_1, h_2\}$) and similarly two L-classes. If $A = \{a, b\}$ the only languages recognized by $\mathcal{J}_3$ are a^ϵ, b^ϵ, aA^ϵ, bA^ϵ, $A^\epsilon a$, $A^\epsilon b$ where $\epsilon \in \{*, +\}$ and their pairwise intersections and unions.

We point out that the above question (characterizing classes of monoids through the properties of the languages they recognize) can be considered also for classes that are not necessarily varieties. The monoid $\mathcal{J}_3$ is a special case of the Jones monoids $\mathcal{J}_n$ generated by h_i, $i = 1, \ldots, n-1$, with relations (A) $h_ih_jh_i = h_i$ for $|i-j| = 1$, (B) $h_ih_i = h_i$ and (C) $h_ih_j = h_jh_i$ for $|i \quad j| \geq 2$. Unfortunately, except for $n = 3$, the Jones monoids are not idempotent monoids, but they are all finite monoids [1, 21]. Given the relations (A)–(C), what are the properties of the languages recognized by $\mathcal{J}_n$? Can those properties be listed such that monoids $\mathcal{J}_n$ can be characterized?

RECOGNIZABLE VS REGULAR

Kleene's theorem says that $\mathcal{R}eg(A^*)$ coincides with the smallest class of languages that contain all finite languages and is closed under union, product and $*$-operation ($L^* = \cup_{i=0}^{\infty} L^i$ where $L^0 = \{1\}$) [13]. Similarly, the smallest class of subsets of M that contains all finite subsets of M, is closed under union, product and $*$-operation is the set of rational M-languages denoted $\mathcal{R}at(M)$. It can be observed that $\mathcal{R}at(M) = \mathcal{R}eg(M)$ for every M [5].

Consider $A = \{a, b\}$ and $M = A^* \times A^*$. A two state automaton (states q_1, q_2 with transitions $q_1(a, 1) = q_2$ and $q_2(1, b) = q_1$, having initial and terminal state q_1) recognizes the M-language $L_M = (a, b)^* = \{(a^n, b^n) \mid n \geq 0\}$. One can add a junk state q_3 sending all other missing transitions with generators $(a, 1), (b, 1), (1, a), (1, b)$ to this state. Hence $(a, b)^* \in \mathcal{R}eg(M)$. Let N be a finite monoid and a morphism $\eta : M \to N$. Let $\eta(a, 1) = x$ and $\eta(1, b) = y$. Because $(a, 1)(1, b) = (1, b)(a, 1) = (a, b)$ we must have $xy = yx$ in N. If $P \subseteq N$ is such that $\eta^{-1}(P) = L_M$, then P must contain the submonoid of N generated by xy. However, due to the commutativity, $(xy)^n = x^ny^n$. Because N is finite, there are n, k such that $x^{n+k} = x^n$ and therefore $\eta(a^{n+k}, b^n) = x^{n+k}y^n = x^ny^n \in P$. But $(a^{n+k}, b^n) \notin L_M$. Thus $L_M \notin \mathcal{R}ec(M)$. Observe that *there is no morphism* from M to the transition monoid of this M-automaton since the action of $(a, 1)(1, b)$ on the states q_1, q_2, q_3 is not the same as the action of $(1, b)(a, 1)$, although they represent the same element in M.

By Proposition 2.1 we have that $\mathcal{R}eg(A^*) = \mathcal{R}ec(A^*)$. The equivalence of two automata, or the emptiness problem for A^*-automata are easily decidable. $\mathcal{R}ec(A^*)$ is closed under intersection, but in general $\mathcal{R}ec(M)$ is not necessarily so, and that poses the main problem in understanding $\mathcal{R}ec(M)$. The simple example above is a base of several undecidability observations.

Proposition 4.1

(a) There is a monoid M such that it is undecidable whether $L_1 \cap L_2 = \emptyset$ for $L_1, L_2 \in \mathcal{R}eg(M)$ [18].

(b) There is a monoid M such that it is undecidable whether two M-automata recognize the same language [6].

In general, for a finitely generated monoid M, $\mathcal{R}ec(M) \subseteq \mathcal{R}eg(M)$ [16], but, as we saw above, the other inclusion does not necessarily hold. If M is not finitely generated, then even this inclusion does not hold. Take $M = \mathbb{Z}$ with the multiplication of numbers as the operation. One can map $\mathbb{Z}$ into a three element monoid $\{0, 1, 2\}$ with 0 being a zero element, 1 being an identity, and $2 \cdot 2 = 0$. The map sends $1, -1 \mapsto 1$, prime $\mapsto 2$, and non-prime $\mapsto 0$. The inverse image of $\{2\}$ is the set of primes (hence the set is recognizable), but it cannot be a regular set since no automaton can recognize this set.

When M is a group we have the following:

Proposition 4.2. [19] *Let G be a finitely generated group and $H \leq G$. Then*

(a) $H \in \mathcal{R}eg(G)$ iff H is finitely generated

(b) $H \in \mathcal{R}ec(G)$ iff H has finite index.

Problem: Characterize Kleene's monoids, i.e., monoids with $\mathcal{R}ec(M) = \mathcal{R}eg(M)$.

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ПРЕПОЗНАТЛИВИ И РЕГУЛАРНИ ПОДМНОЖЕСТВА ОД МОНОИДИ

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Во спомен на професор Ѓорѓи Чупона со длабока благодарност

Овој запис е кратка ревизија на регуларни и препознатливи подмножества од моноиди. Воведуваме ново прашање за карактеризација на класи моноиди и покажуваме дека класата на иденпотентни моноиди може да се карактеризира преку својствата на јазиците препознаени со оваа класа.

Клучни зборови: моноиди, автомати, регуларни јазици, препознатливи јазици

Имав привилегија професор Чупона да ми помогне да го започнам моето професионално патешествие и му должам голема благодарност. Мојата дипломска работа беше на тема комбинаторна теорија на групи и професор Чупона беше мој ментор. Подоцна за време на моите докторски студии се навратив на оваа тема, а и мојата докторска дисертација заврши со добар дел посветен на полугрупи. Како асистент на Институтот по Математика за две-три години, професор Чупона ми ја препорача книгата Теорија на Автомати од Арто Салома [20], и таа област заврши како главна тема на мојата дисертација и истражување ([9, 10, 11]). А повеќе од сè беа моментите на дружење што тој ги креираше и што не правеа блиски со него, а и блиски меѓу нас како другари математичари.